Dividing the Indivisible: Apportionment and Philosophical Theories of Fairness

Abstract

Philosophical theories of fairness propose to divide a good that several individuals have a claim to in proportion to the strength of their respective claims. We suggest that currently, these theories face a dilemma when dealing with a good that is indivisible. On the one hand, theories of fairness that use weighted lotteries are either of limited applicability or fall prey to an objection by Brad Hooker. On the other hand, accounts that do without weighted lotteries fall prey to three fairness paradoxes. We demonstrate that division methods from *apportionment theory*, which has hitherto been ignored by philosophical theories of fairness, can be used to provide fair division for indivisible goods without weighted lotteries and without fairness paradoxes.

Keywords

Fairness; Fair Division; John Broome; Indivisibility; Indivisible Good; Lotteries; Fairness of Lotteries; Apportionment

1 Introduction

Those in charge of allocating a limited amount of an available good often face a problem: how do they ensure that they treat everyone fairly? Answering this question raises many complicated issues even if the good in question is quite easily divisible, such as a piece of cake or fruit, or a large enough amount of money. Yet, there are also goods that are not divisible, such as a kidney transplant.

Perhaps the best known account of fairness that applies to such problems is that of John Broome (1990). His contribution has started a lively philosophical debate about how to be fair. For divisible goods, he recommends to satisfy individual claims in proportion to their strength. For indivisible goods, he recommends using lotteries, with equal or unequal weights, depending on whether claims have equal strengths or not. Broome's account of fairness has been criticised in a variety of ways,[1] yet it is especially Broome's use of *(unequally) weighted lotteries* that has been found wanting.[2]

We examine how current philosophical theories of fairness that have been proposed as developments of Broome's (1990) account fare with regards to indivisible goods. They can be divided into two families of accounts: some recommend the use of weighted lotteries, such as Broome's (1990) original account, and the theory proposed by Curtis (2014). We will show that these accounts either have to stay silent on a number of important cases or fall prey to an objection by Hooker (2005). Other accounts, such as the one by Lazenby (2014), do without weighted lotteries and thus evade these problems, but we will argue that these accounts fall prey to three fairness paradoxes. Analysing fair division of indivisible goods on the basis of current philosophical theories of fairness thus seems to present us with a dilemma.

We argue that *apportionment theory* provides a method that helps to avoid such a dilemma. Apportionment theory (Balinski and Young 2001) is the systematic study of methods that can be and have been used to solve problems of proportional representation that involve indivisible goods (such as seats in parliament). Although the main motivation of apportionment theory has been to study the concrete problem of (fair) political representation, it is applicable to a much wider class of problems, including the division problems as discussed in the philosophical literature on fairness.

Section 2 presents the first horn of the dilemma: theories of fairness that use weighted lotteries are either of limited applicability or fall prey to an objection by Hooker. Section 3 presents the second horn of the dilemma: accounts that do without weighted lotteries fall prey to three fairness paradoxes. Section 4 demonstrates that apportionment theory provides us with division methods for indivisible goods that do not exploit weighted lotteries and that escape the fairness paradoxes: relying on such methods allows one to escape the dilemma. Section 5 briefly concludes.

2 Philosophical theories of fairness and Hooker's objection

We discuss Broome's (1990) account of fair division and a theory of fairness proposed by Curtis (2014). We suggest that the only way in which they can avoid the criticism of Hooker (2005) is by remaining silent on a number of important fair division problems, which is undesirable.

2.1 Broome's theory of fairness

The core feature of Broome's (1990) account may be described by the slogan "fairness requires that claims are satisfied in proportion to their strength", where claims are a certain type of reasons, owed to the agent, as to why an agent should have some of the good. For fair division problems that involve a *divisible* good such as money, the slogan by itself more or less suffices to determine what a fair allocating agent must do. However, to apply Broome's theory to fair division problems with *indivisible* goods, we need to take recourse to notions of the account that go beyond its slogan. Let us illustrate this via the following example.

Kidney. Ann and Bonny both need a kidney transplant in order to survive, but there is only one kidney available. If given to Ann, she gains 25 years of life, while Bonny gains 20 years of life if given the kidney. How should we allocate the kidney?

Suppose that we decide to give the kidney to Ann as, all else being equal, she benefits more—i.e. an extra five years—from receiving the kidney than Bonny: we decide to give it to Ann as she has a stronger "reason of benefit" to get the kidney than Bonny does. Now, although Ann has a stronger reason of benefit than Bonny, we owe it to both women to save their lives: Ann and Bonny have equally strong *claims* to the kidney. As equal claims require equal satisfaction, the *fairest* thing to do, according to Broome, is to destroy the kidney. This is not to say that one should—all things considered—destroy the kidney: fairness is a *pro-tanto* value that has to be balanced against other moral values. This balance dictates that the allocation of the kidney should proceed via an *equally weighted* lottery. By having such a lottery, the equal claims of Ann and Bonny receive equal *surrogate satisfaction*, which ensures a "second-best" kind of fairness. Although the lottery is neither the fairest thing to do (which is to destroy the kidney) nor the best thing to do (which is to give it to Ann), it strikes a balance between two competing moral values in such a way that it is the all-things-considered right thing to do. Or so Broome argues.

Broome's account of fairness may also require allocation via unequally weighted lotteries. For example, suppose that Bonny has an irresponsible, unhealthy lifestyle of a type that is highly correlated with her kidney disease. Further suppose that Ann's kidney disease can only be ascribed to brute bad luck. In that kind of situation, call it Unequal Kidney, Ann arguably has a stronger claim to the kidney than Bonny has. According to Broome, the fairest way to allocate the kidney in Unequal Kidney is to have a weighted lottery in which "ideally, each person's chance should be in proportion to the strength of her claim" (Broome 1990:98).

Broome thus offers a conceptually fruitful theory of fairness that allows one to articulate the role that fairness, understood as a proportional function of claims, plays in our all-things-considered moral judgements. Broome has illustrated his theory in terms of cases such as Kidney and Unequal Kidney, in which a *single* (unit of a) indivisible good has to be divided. However, for cases that involve *multiple* indivisible goods, Broome's theory does not tell us how, in order to be fair, we must divide the amount of available good. More generally, Curtis (2014:47) argues that Broome has not "laid down a theory of precisely what one must do in order to be fair". Curtis (2014) does not leave it at this remark but has ventured to develop Broome's account into a more precise theory of fairness, as discussed in the next section.

2.2 Curtis's theory of fairness

Curtis (2014:47) provides a theory of fairness that aims at making Broome's account more precise:

> . . . I present a theory of what it is to be fair. I take my cue from Broome's well-known 1990 account of fairness. Broome's basic thesis

> is that fairness is *the proportional satisfaction of claims*, and with this I am in at least partial agreement. But neither Broome nor anyone else (so far as I know) has laid down a theory of precisely what one must do in order to be fair. The theory offered here does just this.

Curtis provides a by and large Broomean theory of fairness that precisely specifies what a fair allocating agent must do in situations such as the following.

Apples. Jones possesses 4 apples and has promised 9 apples to Smith, 8 to Brown, and 3 to Clark. So (we may suppose) Smith has a claim to 9 apples and Brown has a claim to 8 apples whilst Clark has a claim to 3 apples.

More generally, Curtis (2014:47) describes the situations to which his theory applies as "occurrences where an *allocating agent* has an amount of a good and allocates it amongst a set of *receiving agents*, each of whom has a *claim* to the good in question". Curtis's theory is applicable to both situations in which the good-to-be-divided is divisible as well as to problems where the good is indivisible. When the good-to-be-divided is divisible, the theory dictates that the allocating agent must adopt *method P*.

Method P. Divide the estate proportional to the agents' claims.

And so, when we assume that the apples in Apples are divisible pieces of fruit, Curtis's theory prescribes that Smith receives $\frac{9}{9+8+3} \cdot 4 = 1.8$ apples whereas Brown receives 1.6 apples and Clark receives 0.6 apples. Note that the theory thus prescribes that *all of the* four apples must be allocated: according to Curtis, fairness requires *efficiency*. At this point, Curtis thus explicitly departs from

Broome's theory.[3]

Things get more interesting with indivisible goods. So let us now assume that the apples in Apples are indivisible computers. In that scenario, Curtis's theory prescribes that Jones must act according to *method L*.

Method L.

- L_1 Calculate the *proportional share* of each agent, which is the amount that the agent would get according to method P if the estate were divisible.
- L_2 Allot each agent the integer part of his proportional share.
- L_3 Allocate the remaining units of the estate via a lottery in which the probability that an agent receives a unit of the estate is equal to the agent's *fractional remainder*, i.e. to the fractional part of his proportional share.

In Apples, the proportional shares of Smith, Brown, and Clark are 1.8, 1.6, and 0.6 apples respectively, which means that step L_2 comes down to allotting to the agents 1, 1, and 0 apples, respectively. Doing so leaves us with $4 - 2 = 2$ apples to be allocated via step L_3, i.e. via a lottery in which the probabilities that Smith, Brown, and Clark receive an apple are, respectively, 0.8, 0.6, and 0.6.

How to set up such a lottery? The outcomes of the lottery are those allocations of the remaining good in which no agent receives more than one unit of it. So in Apples, those allocations are $(1, 1, 0)$, in which only Smith and Brown receive an apple, and $(1, 0, 1)$ as well as $(0, 1, 1)$. Let's call the probabilities of these three outcomes x, y, and z respectively, values of which can be found by solving a system of linear equations which gives us $x = y = 0.4$ and $z = 0.2$. Hence the lottery produces outcomes $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ with probabilities 0.4, 0.4, and 0.2 respectively. In this lottery, the probability that an agent receives an apple is equal to the fractional part of his fair share, as required by L_3. More generally, Curtis shows that a lottery satisfying L_3's description can always be found by solving a system of linear equations and hence that method L is well-defined.

To sum up, Curtis's theory tells us that a fair allocating agent must adopt method P and method L for division problems in which the good-to-be-divided is, respectively, divisible and indivisible.

At first glance, it seems that Curtis's theory of fairness is an important precisification of Broome's account, and that it thus improves on the latter. However, whereas the slogan of Broome's account is that "fairness requires that claims are satisfied in proportion to their *strength*", Curtis (2014:47) writes that "Broome's basic thesis is that fairness is the *proportional satisfaction of claims*" and he thus leaves out the "in accordance with their strength" addendum. Indeed, the notion of the strength of a claim does not figure in Curtis's theory at all. Has Curtis's precisification of Broome's theory left out a crucial aspect of that theory? Yes it has, but there is a natural way to incorporate claim strengths into Curtis's theory, as we will explain in the next section. In doing so, we will also relate the theories of Broome and Curtis to the axiomatic literature in economics that deals with claims problems.

2.3 Amounts and strengths in fair division problems

In Apples, Jones has promised 9 apples to Smith, 8 to Brown, and 3 to Clark. As promises generate claims, we may say that Smith has a claim *to* 9 apples *in virtue of* Jones's promise. Likewise, Clark has a claim *to* 3 apples *in virtue of* Jones's promise. These descriptions are instantiations of the following *claim schema*:

Agent A has a claim to an amount X of a good in virtue of Y.

The amount X of the good to which the agent has a claim, we call the *amount* of the claim. The reason Y that generates the claim we call the *ground* of the claim. Claims with identical grounds have, per definition, equal *strength*. So Smith has a claim to 9 apples of the same strength as Clark's claim to 3 apples. However, claims with different grounds will typically have different strengths. For instance they do so in Unequal Kidney (cf. Section 2.1) but also in the following example, due to Hooker (2005:349).

Medicine. Suppose that there is available some limited quantity of medicine and that this medicine cannot be divided without rendering it ineffective. Suppose Ann's claim to the medicine comes from the fact that she needs it to save her life, and Bonny's claim to it comes from the fact that she needs it to save her little finger. Suppose an average life is something like a thousand times more important than a little finger.

In Medicine, Ann has a claim to the medicine in virtue of the need to save her life whereas Bonny has a claim to the medicine in virtue of the need to save her little finger: the amount of their claims is the same, but the grounds of the claims of Ann and Bonny differ. Clearly Ann's claim to the medicine is much stronger than Bonny's; Hooker assumes that Ann's claim is 1,000 times stronger than Bonny's.

Curtis identifies claims with their amounts and so he, albeit implicitly, assumes that all claims are of equal strength. In other words, Curtis's theory only applies to what we will call *claims problems*.

Definition 1 (Claims problems) A *claims problem* $\mathcal{C} := (E, N, c)$ consists of a (divisible or indivisible) amount of good $E > 0$, also called the *estate*, a set of agents N, and a claims vector c specifying the amount $c_i > 0$ of the estate that agent i has a claim to and which is such that $\sum_{i \in N} c_i \geq E$.

It may seem a bit odd to reserve the label "claims problem" for those fair division problems that only recognise one part of a claim: while claims problems recognise the amount, they neglect the strength of a claim. However, the structures of Definition 1 are also called "claims problems" in the economic literature on fair division. In this literature, claims problems, and the division rules that propose how to divide the estate in such problems, are studied extensively—see Thomson (2003) for an overview. Hence, calling the structures of Definition 1 claims problems is convenient as it makes explicit that Curtis's philosophical theory of fairness applies to problems that have been studied extensively in the economic literature. Actually, the results obtained in this literature have bearing on Curtis's arguments as to *why* fairness requires us to adopt method P and L, as we argue in Heilmann and Wintein (2015).[4]

Claims problems only allow us to specify the amount of a claim. Thus, fair division problems such as Medicine, in which claim strengths play a crucial role, do not give rise to claims problems. We will say that Medicine gives rise to a *Broomean problem.*

Definition 2 (Broomean problems) A *Broomean problem* $\mathcal{B}$ is given by a tuple (E, N, c, s) where (E, N, c) is a claims problem and where s is a strengths vector specifying the strength $s_i > 0$ of the claim of agent i.

As Curtis's theory of fairness only applies to claims problems it has, strictly speaking, nothing to say about Broomean problems such as Medicine.[5] This is rather unsatisfactory, since any theory of fairness should tell us what a fair allocating agent must do in Broomean problems such as Medicine. However, this unsatisfactory aspect of his theory can be removed as there is a natural way to *incorporate* claim strengths into his theory, so that Curtis's theory becomes applicable to Broomean problems. We will first specify *how* we incorporate claim strengths into Curtis's theory and thereafter explain *why* they must be incorporated in this manner.

Given a Broomean problem $\mathcal{B} = (E, N, c, s)$, the product of the amount c_i and strength s_i of agent i is called the *strength-corrected claim* of that agent. Let us write r for the vector of strength-corrected claims, so that $r_i = s_i \cdot c_i$ records the strength-corrected claim of agent i. Note that the triple (E, N, r) satisfies the formal definition of a claim problems except for the fact that the sum of r_i does not necessarily exceed the estate E. Also, whereas the entries of the vector c of a claims problem are specified in the same currency as the estate (e.g. apples), this does not hold for the vector r of strength-corrected claims. We will call structures such as (E, N, r) *corrected claims problems*.

Definition 3 (Corrected claims problems) A *corrected claims problem* $\mathcal{C} := (E, N, r)$ consists of a (divisible or indivisible) estate $E > 0$, a set of agents N, and a vector r specifying the strength-corrected claim $r_i > 0$ of agent i.

As the reader may care to verify, the close similarity between claims problems and corrected claims problems ensures that method P and method L are readily applicable to corrected claims problems. Hence, Curtis's theory can be applied to divide the estate in a Broomean problem in the following way.

Curtis's Broomean method. Let $\mathcal{B} = (E, N, c, s)$ be a Broomean problem. In order to divide the estate E of $\mathcal{B}$ via Curtis's (extended) theory, one proceeds as follows.

1. Translate $\mathcal{B}$ as corrected claims problem $\mathsf{tr}(\mathcal{B}) = (E, N, r)$, where $r_i = s_i \cdot c_i$.
2. If E is divisible, divide E by applying method P to $\mathsf{tr}(\mathcal{B})$;

If E is indivisible, divide E by applying method L to $\mathsf{tr}(\mathcal{B})$.

Curtis's Broomean method spells out *how* to incorporate claims strengths into Curtis's theory. Let us now turn to the rationale of this method, which to a large extent follows from the following proposition.

Proposition 1 *Let* $\mathcal{B} = (E, N, c, s)$ *be a divisible Broomean problem and suppose that agent* j *has a claim that is* $\lambda > 0$ *times as strong as agent* i*'s claim:* $s_j = \lambda \cdot s_i$*. Let* x_j *and* x_i *be the amounts that are allotted to* j *and* i*, respectively, by applying method* P *to* $\mathsf{tr}(\mathcal{B})$*, i.e. to the corrected claims problem that is obtained as the translation of* $\mathcal{B}$*. Then* j*'s claim receives* λ *times as much (percentage-wise) satisfaction as* i*'s claim:*

$$\frac{x_j}{c_j} \cdot 100\% = \lambda \cdot \frac{x_i}{c_i} \cdot 100\%$$

Proof: By an inspection of definitions. □

According to the Broomean slogan, claims must be satisfied in proportion to their strength. Proposition 1 testifies that Curtis's Broomean method respects this slogan: when agent j's claim is λ times as strong as agent i's claim, her claim receives λ times as much satisfaction. Curtis's (original) theory is intended to be a precisification of Broome's theory. Hence, Proposition 1 testifies that our incorporation of claim strengths into Curtis's theory is a natural one, as it respects this intention. Proposition 1 is only concerned with divisible Broomean problems and hence with method P. However, as method L is basically an extension of method P to the indivisible case (as an inspection of step L_1 and L_2 of method L reveals), our incorporation of claim strengths also qualifies as natural when we take indivisible Broomean problems into account.[6]

By a *broadly Broomean theory of fairness* we mean any theory of fairness that subscribes to the Broomean slogan that fairness requires that claims are satisfied in proportion to their strength. We take it that any such theory must divide the estate E in a *divisible* Broomean problem $\mathcal{B}$ in proportion to the strength-corrected claims of the agents. And so this division of E can be obtained by applying method P to corrected claims problem $\mathsf{tr}(\mathcal{B})$. More generally, we will assume that for any broadly Broomean theory of fairness, it suffices to specify how it proposes to allocate the estate in corrected claims problems: such theories can rely on the translation tr to describe their proposals for Broomean problems.

Thus, broadly Broomean theories of fairness only differ in the method that they propose to allocate the estate for corrected claims problems with an indivisible estate. Curtis's extended theory is a broadly Broomean theory that proposes to divide indivisible estates via method L, which typically results in holding a weighted lottery. We now turn to an objection to the use of such lotteries for fairly dividing a good and so by extension to Broome's and Curtis's extended theory.

2.4 Hooker's objection

Let us revisit Medicine, the Broomean problem that was introduced in Section 2.3. In terms of the notation of Definition 2, Medicine is represented as follows.

$$\mathcal{B}^{\mathsf{Med}} = (1, \{A, B\}, (1, 1), (999, 1))$$

Remember that according to Broome, in a case where a single indivisible good has to be divided amongst agents with claims to the good of unequal strength, fairness requires that "ideally, each person's chance should be in proportion to the strength of her claim" (Broome 1990:98). As Medicine is such a case, who gets the medicine should, according to Broome, be decided by a lottery in which Ann has a chance of winning of 999/1000 and Bonny has a chance of 1/1000. The same weighted lottery is prescribed by Curtis's extended theory, as the reader may care to verify by applying method L to $\mathsf{tr}(\mathcal{B}^{\mathsf{Med}})$.

Hooker objects to the use of weighted lotteries in Medicine, as having such a lottery means that we risk that Ann dies, as she might not end up with the medicine. As Hooker (2005:349) puts it, with respect to cases like Medicine, "letting the stronger claim win seems completely fair". Hooker is far from alone in his judgement that in cases like Medicine, fairness requires that the good goes to the agent with the strongest claim. Other commentators on Broome's theory, such as Lazenby (2014), Kirkpatrick and Eastwood (2015), and Piller (2016) all agree with Hooker on this point.[7] These shared *intuitions* concerning cases like Medicine make it worthwhile to develop a *theory* of fairness according to which, indeed, it is completely fair to let the stronger claim win.

In addition to Broome's theory, also Curtis's extended theory of fairness falls prey to Hooker's objection. Now of course Curtis could resist an extension of his theory to account for claim strengths. But doing so would leave him with his official theory that simply has to remain silent, not just on Medicine, but on *any* Broomean problem. This silence would then motivate the development of an extension of Curtis's theory to cover Broomean problems, as we did in Section 2.3. The fact that the extended theory falls prey to Hooker's objection then, again, motivates the development of a theory of fairness that prescribes that in cases like Medicine "the stronger claim should win". We will now turn to such a theory.

3 Lazenby's account and three fairness paradoxes

We have just established the first horn of our dilemma for philosophical theories of fairness, which showed the problems and limitations of theories that use weighted lotteries. We now turn to an account of fairness by Hugh Lazenby (2014), who proposes to analyse fair division of indivisible goods without weighted lotteries. Although his proposal is interesting, it is also rather imprecise. However, in Section 3.1 we argue that there is a canonical way in which his account can be made precise. In Section 3.2, we then argue that Lazenby's theory of fairness thus obtained falls prey to three fairness paradoxes. We further discuss and dismiss a potential objection—regarding an assumption of efficiency—to our precisification. Doing so establishes the second horn of the dilemma for philosophical theories of fairness.

3.1 Lazenby's account and the largest remainder method

Recently, Lazenby (2014) has articulated a broadly Broomean conception of fairness with one notable difference: according to Lazenby (2014:331), "any contribution to fairness from entering claims into a lottery is lexically posterior to fairness in outcome". As a consequence, letting the stronger claim win in cases like Medicine *is* completely fair, and so Hooker's intuition becomes a feature of Lazenby's account of fairness. Moreover, on Lazenby's account, cases like Kidney can receive the same appealing Broomean analysis that we sketched above. And so, Lazenby's account of fairness seems rather attractive, as it manages to keep the most appealing features of Broome's account whilst getting rid of its most controversial one.

Lazenby argues for his central thesis, i.e. the claim that "lottery fairness is lexically posterior to outcome fairness" by scrutinising the relation between *surrogate satisfaction*, which claim holders receive by having their claims entered into a lottery, and *outcome satisfaction*, which claim holders receive by having some good in outcome. Although Lazenby's central thesis is clearly an attractive one, it needs to be said that Lazenby does very little to indicate how his thesis plays out in concrete fair division problems. In fact, Lazenby only discusses two examples of fair division problems to illustrate his account of fairness. One of these examples is Unequal Kidney, the other is the Dillon & Emma example, to be discussed below. As Lazenby seeks to present a broadly Broomean account of fairness (cf. section 2.3) we want to know how his account proposes to allocate the estate in an arbitrary (indivisible) corrected claims problem. We will argue below that, given what Lazenby tells us about Unequal Kidney and Dillon & Emma, it is natural to associate his account with *the largest remainder method*, which proposes to allocate the estate in an (indivisible) corrected claims problems as follows.

The largest remainder method. First give to each agent the integer part of his *proportional share* (cf. section 2.2). If not all of the estate is allocated, give one unit of the remaining estate to each of the agents with the largest fractional remainders. If agents having the same fractional remainder are candidates for receiving a unit of the remaining estate, decide who gets the unit by holding an equally weighted lottery amongst those agents.

Let us first illustrate the largest remainder method by considering indivisible Apples. The proportional shares of Smith, Brown, and Clark are 1.8, 1.6, and 0.6 respectively. And so, allotting the integer parts of these shares comes down to giving an apple to both Smith and Brown. The remaining two apples are then, according to the largest remainder method, to be allotted as follows. First, an apple is given to Smith as he has the largest fractional remainder. Second, as both Brown and Clark have the second largest remainder, an equally weighted lottery should decide whether Brown or Clark gets the last apple. Hence, applying the largest remainder method to (indivisible) Apples comes down to holding an equally weighted lottery with allocations $(2, 2, 0)$ and $(2, 1, 1)$ as outcomes.

Let us now explain why, given what Lazenby tells us about Unequal Kidney and Dillon & Emma respectively, it is natural to associate his account with the largest remainder method.

With respect to Unequal Kidney, Lazenby (2014:331) remarks that, in comparison with having a weighted lottery, "it is fairer to give the kidney directly to Ann as her claim is strongest". Not only is this verdict in agreement with the prescription of the largest remainder method for Unequal Kidney, but the largest remainder method also naturally generalises Lazenby's remark that Ann should have the kidney as *her claim is strongest*. To see this, we first observe that in cases where the estate consists of more than a single unit, it seems uncontroversial that each agent should receive at least the integer part of his proportional share. Indeed, for a broadly Broomean theory of fairness, the vector induced by proportional shares represents the "fairest division of the estate" which, however, is unattainable when dealing with an indivisible estate: some agents must receive more than their proportional share while others must receive less. However, it seems reasonable that no agent receives less than the integer part of his proportional share. For as it is always possible to give each agent this integer part, an allocation in which an agent receives less means an unnecessary large deviation from the fairest division. Second, the fractional remainder of an agent is readily interpreted as the strength of his claim to receive one unit of the *remaining estate*, i.e. the estate that remains when all agents have received the integer part of their proportional share. And so, the largest remainder method proposes to allocate the remaining estate "according to the strongest claims" which is clearly in line with Lazenby's remarks on Unequal Kidney.

Lazenby's discussion (2014:335) of Dillon & Emma is much more elaborate, and also more informative, than his discussion of Unequal Kidney.

Dillon & Emma. Assume that we have two otherwise equally situated individuals, Dillon and Emma. Dillon has a claim of strength 6 and Emma has a claim of strength 4. The value of the good to be divided between them is 100.

Although Lazenby does not explicitly specify what Dillon and Emma have a claim to, the amount of their claims seems most aptly described by saying that they have a claim to (a good with) a value of 100. At any rate—given that Lazenby's account is broadly Broomean—the relevant structure of Lazenby's example is aptly described as the divisible corrected claims problem $\mathcal{C}^{\mathsf{Laz}} = (100, \{\text{Dillon}, \text{Emma}\}, (6, 4))$. Lazenby argues that his account leads to the following ranking, in terms of what is fairest to do, of some possible courses of action in his example:

1. Give 60 to Dillon and 40 to Emma.
2. Destroy the good.
3. Give 100 to Dillon.
4. A weighted lottery—60 % chance of 100 to Dillon, 40 % chance of 100 to Emma.
5. Toss a coin—50 % chance of 100 to either.
6. Give 100 to Emma.

Here is how Lazenby argues that his account induces this ranking:

> Dividing the good in proportion to the agents' claims is the most fair. Destroying the good is more fair than giving the good directly to Dillon because it satisfies claims more proportionately in outcome. Giving the good directly to Dillon is more fair than holding a weighted lottery since it has a better expected fairness in outcome. A weighted lottery is more fair than a lottery at equal odds since it provides a more proportionate surrogate satisfaction of each claim. Giving the good directly to Emma is the least fair since there is no

lottery and claims are least proportionately satisfied in outcome.

Lazenby (2014:336)

For our purposes, all that matters is that the above quote clearly indicates that Lazenby ranks allocations of the estate according to the extent to which they satisfy the claims proportionally in outcome: allocations that do so to a greater extent are fairer than those that do so to a lesser extent. But how then do we precisely measure to which extent an allocation satisfies claims in proportion to their strength? A natural way to make precise Lazenby's remarks on allocations that provide more or less proportional satisfaction in outcome is in terms of the (Euclidean) distance $d(x,p)$ between an allocation x and the "fairest division" as given by the vector of proportional shares p, where

$$d(x,p) = \sqrt{\sum (x_i - p_i)^2}$$

On the proposal under consideration, the smaller the distance $d(x,p)$, the more allocation x satisfies claims in proportion to their strength and thus the fairer the allocation. The motivation that Lazenby provides for his ranking thus suggests that, when made precise, his account prescribes the following allocation method for indivisible corrected claims problems.

Least distance method. Divide the estate by picking the allocation x that minimises $d(x,p)$, i.e. the distance to the vector of proportional shares. If several allocations minimise this distance, pick one of those allocations by holding an equally weighted lottery.

In order to illustrate this method, Table 1 lists all 15 possible allocations of the 4 apples in (indivisible) Apples, where an allocation $x = (x_S, x_B, x_C)$ allots x_S, x_B, and x_C apples to Smith, Brown, and Clark respectively. The 15 possible allocations correspond to the columns of Table 1 and the last entry of such a column indicates the distance (rounded to one decimal place) of the allocation to the vector of proportional shares $p = (1.8,\ 1.6,\ 0.6)$ of Apples.

x_S	4	0	0	3	3	1	1	0	0	2	2	0	2	1	1
x_B	0	4	0	1	0	3	0	3	1	2	0	2	1	2	1
x_C	0	0	4	0	1	0	3	1	3	0	2	2	1	1	2
$d(x,p)$	2.8	3.1	4.2	1.5	2.0	1.7	3.0	2.3	3.1	0.7	2.1	2.3	0.7	1.0	1.7

Table 1: The 15 apple allocations and their distance to $p = (1.8,\ 1.6,\ 0.6)$

As indicated by Table 1, the allocations $(2,2,0)$ and $(2,1,1)$ both minimize the distance to the vector of proportional shares. Hence, applying the least distance method to (indivisible) Apples, comes down to holding an equally weighted lottery with allocations $(2,2,0)$ and $(2,1,1)$ as outcomes.

As the reader may have noticed, applying the least distance method to Apples delivers the same result as applying the largest remainder method to Apples. The following proposition attests that this result holds in full generality and also, it indicates to what extent the result is independent from taking the *Euclidean distance* as one's measure of the amount of proportional satisfaction that is induced by an allocation.

Proposition 2 *For any (indivisible) corrected claims problem $\mathcal{C}$: applying the largest remainder method to $\mathcal{C}$ delivers the same result as applying the least distance method to $\mathcal{C}$. Moreover, the largest remainder method does not only minimise the Euclidean distance between an allocation x and the vector of proportional shares p, but in fact it minimises any ℓ_p norm on $x - p$, such as $\sum | x - p |$.*

Proof: See Birkhoff (1976). □

The upshot of Proposition 2 is as follows. The largest remainder method arguably makes precise Lazenby's analysis of cases like Unequal Kidney for which he proposes to allot the estate to agents with the strongest claims. On the other hand, the least distance method arguably makes precise Lazenby's analysis of the Dillon & Emma example, in which he compares allocations in terms of how much they proportionally satisfy claims in outcome. Proposition 2 tells us that both methods, which seem to capture different aspects of Lazenby's account of fairness, come down to the same thing. And so, Proposition 2 provides evidence for the claim that the largest remainder method can properly serve as a precisification of Lazenby's account of fairness.

With a precise articulation of Lazenby's account of fairness in place, the road is open to a proper assessment of the account: does an agent who exploits the largest remainder method to allocate a limited amount of available good act fairly? We argue that the answer to that question is an unequivocal "no" and put forward an alternative allocation method that is also congenial to Lazenby's remark that "any contribution to fairness from entering claims into a lottery is lexically posterior to fairness in outcome" and that, as we will argue, is fairer than the largest remainder method. But before we turn to that, let us first point out what is wrong with the largest remainder method from the perspective of fairness.

3.2 Three paradoxes for the largest remainder method

More than one thing is wrong with the largest remainder method from the perspective of fairness: it suffers from *three fairness paradoxes*. We will illustrate these paradoxes via examples. All these examples give rise to (genuine) claims problems: the strengths of all claims are the same so that claims can be identified with their amount.

(i) *The more-good-less-satisfaction paradox.*

To illustrate this paradox, consider the following example.

Horses. An old landlord owns 13 horses. In writing his will he specifies that (and only specifies that) upon his death, 6 of them should go to Alice, 6 to Bob, and 1 to Charlie. A day before the landlord's death, 8 of his horses are stolen so that the executer of his will must decide how to allocate the remaining 5 horses.

Suppose that the executer allocates the horses via the largest remainder method. On Monday, the executer announces to Alice, Bob, and Charlie that they will receive 2, 2, and 1 horses, respectively, and that the horses will be delivered to them within one week's time. However, on Tuesday one of the stolen horses is returned and so the executer has to reconsider the proposed allocation as there are now 6 horses to be divided. The executer recalculates the largest remainder allocation and finds that according to this allocation Alice, Bob, and Charlie receive 3, 3, and 0 horses, respectively. Indeed, in the novel situation there is more good to be allocated and yet Charlie receives less of it: the largest remainder method suffers from the *more-good-less-satisfaction paradox*. When the executer informs Charlie about the returned horse and explains that Charlie does not, in contrast to the earlier announcement, receive a horse, Charlie first looks puzzled. Then, when Charlie realises that the executer is not joking, he shouts: "but that's not fair!" And right he is.

(ii) *The leaving-claimant paradox.*

To illustrate this paradox, consider the following variant of Horses.

More Horses. An old landlord owns 105 horses. In writing his will he specifies that (and only specifies that) upon his death, 11 of them should go to Alice, 89 to Bob, and 5 to Charlie. A day before the landlord's death, 90 of his horses are stolen so that the executer of his will must decide how to allocate the remaining 15 horses.

Suppose that the executer allocates the 15 horses via the largest remainder method, according to which Alice, Bob, and Charlie should receive 1, 13, and 1 horses, respectively. This time, the executer announces to the heirs that they may pick up their horses at the landlord's house. Charlie is the first to arrive and the executer hands him a horse. Charlie leaves and the executer awaits with the remaining 14 horses for the arrival of the other two heirs. The next to arrive is Alice. When the executer wants to hand Alice her horse, she starts to complain as follows. "There are 14 horses", she says, "and Bob and I have claims of 89 and 11 horses, respectively. And so, as you allocate via the largest remainder method, I should get 2 of the horses and Bob the other 12: just do the math!" After doing the math, the executer finds out that Alice is correct: when 14 horses are to be allocated on the basis of the claims of Alice and Bob, the largest remainder method indeed prescribes that 2 horses should go to Alice and 12 to Bob. On the other hand, the executer's previous calculations of the largest remainder allocation of the 15 horses were also correct, and according to this allocation, Alice should get 1 and Bob 13 horses. The executer is puzzled, as to him it seems that *every part of a fair allocation must be fair.* As the largest remainder method does not (necessarily) result in allocations which have this property, the executer seriously begins to doubt the fairness of this method. And that is exactly what he should do.

(iii) *The claims paradox.*

To illustrate this paradox, consider the following example.

Horse Rentals. Alice, Bob, and Charlie want to go horse riding next Saturday and for that they all want to rent horses at HRENTAL. Alice wants to do so together with 5 friends and so she reserves 6 horses for Saturday. Bob also reserves 6 horses and Charlie, who wants to go on his own, reserves 1 horse. Charlie is the first to arrive at HRENTAL on Saturday and when he does the manager tells him that he has made a bookkeeping mistake. Only 5 horses are available for this Saturday, whereas he has confirmed reservations by Alice, Bob, and Charlie of 6, 6, and 1 horses, respectively. The manager must decide how to allocate the available 5 horses.

Suppose that the manager decides to allocate the 5 horses via the largest remainder method on the basis of the claims of Alice, Bob, and Charlie that are incurred by their reservations. The manager calculates the largest remainder allocation and informs Charlie, who arrived first, that he will get 1 horse whereas Alice and Bob will get 2 horses each. When Alice and Bob arrive at HRENTAL they immediately tell the manager that their reservations have to be adjusted as, due to sickness of some of their friends, Alice and Bob only need 5 and 3 horses, respectively. The manager informs Alice and Bob about his bookkeeping mistake and tells Charlie that he will rerun his calculation in terms of the corrected claims and allocate the 5 horses accordingly. The manager calculates the largest remainder allocation of the 5 horses with respect to the claims of 5,3, and 1 and finds that Alice, Bob, and Charlie should get 3, 2, and 0 horses respectively. When Charlie is informed that the corrected calculations show that he does not get a horse, he is puzzled and reacts as follows. "So Alice's corrected claim is *lower* than her original claim whereas my corrected claim coincides with my original claim. However, the net effect of her having this lower claim is that she gets an *additional* horse that should be allotted to me on the basis of the original claims. That seems to be rather unfair." We concur with Charlie: when the claim of one agent is lowered whereas the claim of another agent is not, it is unfair that the effects of these adjustments are that the first agent receives more of the good whereas the second agent receives less of it. Yet, the largest remainder method allows for unfairness of this kind, as the largest remainder method is subjected to the *claims paradox*.

And so, although the largest remainder method is naturally implied by remarks that Lazenby makes on behalf of his account of fairness, the method is subject to the above three *fairness paradoxes* which makes it hard to qualify it as method of *fair* allocation.

In our precisification of Lazenby's account as the largest remainder method, we have made the assumption of efficiency: all of the available good has to be allocated. Is that a problematic assumption to make, given that Lazenby proposes a strictly comparative account of fairness according to which fairness does *not* require efficiency? We argue no, as the problematic character of the fairness paradoxes for Lazenby's theory of fairness is fully independent of whether or not efficiency is assumed. The most important reason for this is that the division problems that make up the fairness paradoxes, and in fact most real-world cases of fair division, simply ask for a fair division of *all* of the good that is available. For instance, the executer of a testament is (legally) committed to divide all of the heritage amongst the heirs. The efficiency assumption is thus part of the context in which the fairness paradoxes arise. Now the largest remainder method describes the recommendations of Lazenby's strictly comparative account of fairness for those contexts that require efficiency. Hence, the fairness paradoxes show Lazenby's strictly comparative account of fairness to be problematic. For our purposes, the efficiency assumption is thus completely innocuous.

Moreover, when dropping the efficiency assumption, the question as to what, in order to be fair, an allocating agent must do according to Lazenby (or Broome) is arguably not of much interest. To explain why not, note that on a strictly comparative account of fairness, fairness requires, *and only* requires, that claims are satisfied in proportion to their strength. Now consider (in)divisible Apples again and suppose that Jones allots 0 apples to Smith, 0 apples to Brown, and 0 apples to Clark: indeed Jones allots no apple at all. According to a strictly comparative account of fairness, Jones has acted perfectly fair as by allotting no apple at all the equally strong claims of Smith, Brown and Clark receive equal satisfaction (of 0 % each). What holds for Apples holds in full generality: allotting nothing at all is perfectly fair according to a strictly comparative account.[8]

Where does this leave us? We have established the second horn of the dilemma for philosophical theories of fairness: currently existing theories that do not rely on weighted lotteries run into three fairness paradoxes. But we have not yet exhausted all ways to get out of the dilemma: while the paradoxes present problems for Lazenby's theory, they do not disqualify the core idea of Lazenby's account, which is that "any contribution to fairness from entering claims into a lottery is lexically posterior to fairness in outcome". For this core idea can be realised by other allocation methods than the largest remainder method. And so, an interesting question to ask is whether there is an allocation method that captures the core feature of Lazenby's account and that does not fall prey to the three fairness paradoxes. In the next section we answer this question affirmatively.

4 Resolving the dilemma with apportionment theory

We introduce apportionment theory, which is used to solve problems of proportional representation that involve indivisible goods (such as seats in parliament). Although the main motivation of apportionment theory has been to study the concrete problem of (fair) political representation, it is applicable to a much wider class of problems, including the (corrected) claims problems and Broomean problems that play a central role in the philosophical literature on fairness. We show that *Webster's method*, an allocation method studied by apportionment theory, avoids the three paradoxes introduced in Section 3, and since it also does not make use of weighted lotteries, it provides the right tools to resolve the dilemma for philosophical theories of fairness.

4.1 Apportionment theory

Apportionment was first formally studied in relation to political representation. The Constitution of the United States (article I, section 2) states that representatives and direct taxes shall be *apportioned*, i.e. divided proportionally, according to the respective populations of the states. As money is a divisible good, the text of the Constitution unequivocally determines how a total tax burden of T dollars is to be allocated amongst the states: with P_{USA} the total population of the United States and with P_S the population of state S, state S has to pay $\frac{P_S}{P_{USA}} \cdot T$ dollars. However, representatives are indivisible and the text of the Constitution fails to specify what to do with the fractions that result when the proportional formula is applied to allocate the number of seats in the House of Representatives amongst the states according to their populations. Historically, a variety of methods have been used to overcome this *apportion-*

ment problem and the question as to which method best captures the text of the Constitution has been hotly debated.[9]

Apportionment theory is the systematic study of methods that can be and have been used to solve the apportionment problem. Although the main motivation of apportionment theory is to study the concrete problem of (fair) political representation, it is important to realize that apportionment theory is applicable to a much wider class of problems:

> Any problem in which objects are to be allocated in non-negative integers proportionally to some numerical criterion belongs to this class, and [apportionment] theory applies to it (Balinski and Young 2001:96).

Indeed, apportionment theory applies to all problems with the following formal structure.

Definition 4 (Apportionment problems) An *apportionment problem* $\mathcal{A} := (E, N, \rho)$ consists of an indivisible amount of good $E > 0$, also called the *estate*, a set of agents N, and a vector ρ specifying a *record* $\rho_i > 0$ for each agent i.

The problem in an apportionment problem is to *apportion*, i.e. to allocate proportionally, the estate amongst the agents according to their *records*. Those records are simply positive numbers, one for each agent, that are used as the basis for the apportionment. The canonical example of an apportionment problem is a (political) *representation problem*, in which the agents are states, their records are their populations and the estate consists of the available seats in parliament. But as the reader will already have noticed, another example of an apportionment problem is given by an indivisible corrected claims problem $\mathcal{C} = (E, N, r)$. As indivisible corrected claims problems are apportionment problems, one can naturally expect the results of apportionment theory to be relevant for broadly Broomean theories of fairness. And indeed, this turns out to be the case.

In their book *Fair Representation*, Balinski and Young (2001) not only describe how the representation problem has been dealt with in U.S. history, but also present a systematic account of apportionment theory. Various *allocation methods*, which prescribe how to divide the estate in an apportionment problem, i.e. how to solve an apportionment problem, are studied. Although the largest remainder method is a computationally simple allocation method which seems quite fair at a first glance, Balinski and Young dismiss it as a fair allocation method because it falls prey to *the Alabamba paradox*, *the new states paradox*, and *the population paradox*. These three paradoxes, which pertain to the political representation problem, are formally equivalent to the three fairness paradoxes, pertaining to claims problems, that we discussed in the previous section. Balinski and Young not only dismiss the largest remainder method, but also discuss various other allocation methods that do not fall prey to the three paradoxes and which, so they argue, are preferable to the largest remainder method. Moreover, they argue that in particular one such method, *Webster's method*, stands out as *the* method of fair allocation. Webster's method is an example of a so-called *divisor method*, a specific type of allocation method with the following characteristics.

Divisor methods resemble one another in that they all propose to allocate the estate E in terms of a *divisor*. Given a divisor $d > 0$ the *d-proportional share* c_i/d of agent i is equal to his claim divided by the divisor. The *standard divisor*, call it σ, is equal to the sum of all claims divided by the estate, from which it follows that the σ-proportional share of an agent equals his (ordinary) proportional share: $c_i/\sigma = p_i$. Since the estate consist of an indivisible amount of good, these shares have to be rounded. Divisor methods employ specific *rounding rules* to do so. However, the rounded proportional shares typically do not sum to the estate and so these rounded shares do not give rise to an efficient allocation. A divisor method now seeks to obtain an efficient allocation by adjusting the divisor d such that the sum of the *rounded* d-proportional shares of all agents does equal the estate. Divisor methods are thus distinguished from one another by the *rounding rules* that they propose. The rounding rule that is proposed by Webster's method ("Webster rounding") is closely related to "ordinary rounding", with the proviso that when the fractional part of some number r is 0.5, one may round r either up or down. Thus, when we write $[r]_W$ for the set of integers that may result from Webster rounding r, we have for instance $[2.3]_W = \{2\}$, $[2.7]_W = \{3\}$, and $[2.5]_W = \{2,3\}$. Let us now explain how Webster's method proposes to allocate the estate in indivisible Apples.

Webster's Apples. Consider indivisible Apples again. Let us first set our divisor equal to the standard divisor $\sigma = (9+8+3)/4 = 5$. By calculating the Webster rounded σ-proportional shares, we obtain $[9/5]_W = \{2\}$, $[8/5]_W = \{2\}$, and $[3/5]_W = \{1\}$. As $2 \in [9/5]_W, 2 \in [8/5]_W$, and $1 \in [3/5]_W$, the allocation corresponding to the standard divisor equals $(2,2,1)$ which allots one apple more than the estate. We may try to reach an efficient allocation by taking a divisor $d > 5$, which results in lower Webster rounded d-proportional shares. By setting $d = 6$, we get that $[9/6]_W = \{1,2\}$, $[8/6]_W = \{1\}$, and $[3/6]_W = \{0,1\}$. Now $2 \in [9/6]_W, 1 \in [8/6]_W$, and $1 \in [3/6]_W$ which gives the efficient allocation $(2,1,1)$. As $(2,1,1)$ results from Webster rounding (with $d = 6$) and as it is efficient, we say that $(2,1,1)$ is *Webster feasible*. As the only Webster feasible allocation (as one readily verifies) is $(2,1,1)$, Webster's method recommends allocation $(2,1,1)$ for indivisible Apples.

Let us now describe Webster's method precisely. Given an indivisible (corrected) claims problem $\mathcal{C} = (E, N, c)$, the set of all *Webster feasible allocations* is defined as follows.[10]

$$\mathbf{F}_W(\mathcal{C}) = \{x \mid x_i \in [c_i/d]_W \text{ and } \sum x_i = E \text{ for some } d\} \tag{1}$$

Now Webster's method proposes to allocate the estate of an indivisible (corrected) claims problem $\mathcal{C}$ as follows.

Webster's method. When $\mathbf{F}_W(\mathcal{C})$ consists of a single allocation, divide the estate according to that allocation. When $\mathbf{F}_W(\mathcal{C})$ consists of multiple allocations, hold an equally weighted lottery which has those allocations as outcomes.

To illustrate that Webster's method may prescribe to have an *equally* weighted lottery, consider the recommendation of Webster's method for Horses.

Webster's Horses. Consider Horses again. Let us first compute the standard divisor: $\sigma = (6 + 6 + 1)/5 = 2.6$. As $[6/2.6]_W = \{2\}$, $[6/2.6]_W = \{2\}$, and $[1/2.6]_W = \{0\}$, the allocation that is induced by the standard divisor allots too little. To obtain a feasible allocation, we must thus have a divisor d that is lower than 2.6. Setting $d = 2.4$ works, as the reader may care to verify. Indeed, we have $[6/2.4]_W = \{2, 3\}$, $[6/2.4]_W = \{2, 3\}$, and $[1/2.4]_W = \{0\}$ with both $(2, 3, 0)$ and $(3, 2, 0)$ as feasible allocations. Hence, Webster's method prescribes to have an equally weighted lottery with allocations $(2, 3, 0)$ and $(3, 2, 0)$ as outcomes.

For cases like Unequal Kidney or Medicine, Webster's method prescribes that the estate has to go to the claimant with the strongest claim. Indeed, as Webster's method never prescribes to hold an (unequally) weighted lottery, it avoids

Hooker's objection. Moreover, Webster's method does *not* fall prey to the three fairness paradoxes, as attested by the following theorem.

Theorem 1 (Webster's method avoids the three fairness paradoxes)

(1) Webster's method avoids the more-good-less-satisfaction paradox.

(2) Webster's method avoids the leaving-claimant paradox.

(3) Webster's method avoids the claims paradox.

Proof: See Balinksi and Young (2001:70). □

The upshot of Theorem 1 is this: Webster's method is applicable to all corrected claims problems, does not fall prey to Hooker's objection, and avoids the three fairness paradoxes. In other words, Webster's method can be used to escape the dilemma for philosophical theories of fairness that we described earlier.

In fact, Webster's method is not the only allocation method that allows for such an escape. Balinski and Young show that any divisor method avoids the three fairness paradoxes and conversely, that the three fairness paradoxes can only be avoided by a divisor method. From this result it readily follows that the largest remainder method is not a divisor method and also, that all divisor methods can be used to escape the "fairness dilemma". In the next section, we will briefly explain why Webster's method is arguably the "fairest" of these divisor methods.

4.2 Apportionment, bias, and fair division

Remember that divisor methods are distinguished by the *rounding rules* that they exploit. Webster's method exploits the Webster rounding rule, which is close to ordinary rounding, but many more rounding rules exist, such as:

Adams rounding. If r is an integer round r either to r or to $r + 1$.
If r is not an integer round r to the least integer $\geq r$.

Jefferson rounding. If r is an integer round r either to $\max\{0, r-1\}$ or to r. If r is not an integer round r to the greatest integer $\leq r$.

Given such a rounding rule $\mathcal{R}$ and writing $[r]_{\mathcal{R}}$ for the set of integers that may result from rounding r according to rounding rule $\mathcal{R}$, the set of all $\mathcal{R}$ feasible allocations is defined completely similar to (1):

$$\mathbf{F}_{\mathcal{R}}(\mathcal{C}) = \{x \mid x_i \in [c_i/d]_{\mathcal{R}} \text{ and } \sum x_i = E \quad \text{for some } d\},$$

and the definition of *the divisor method based on* $\mathcal{R}$ should not come as a surprise:

The divisor method based on $\mathcal{R}$. When $\mathbf{F}_{\mathcal{R}}(\mathcal{C})$ consists of a single allocation, divide the estate according to that allocation. When $\mathbf{F}_{\mathcal{R}}(\mathcal{C})$ consists of multiple allocations, hold an equally weighted lottery which has those allocations as outcomes.

Interestingly, Balinski and Young prove the following theorem.

Theorem 2 (An allocation method avoids all three fairness paradoxes if and only if it is a divisor method)

Proof: See Balinski and Young (2001:70). □

It follows from Theorem 2 that Jefferson's and Adams's (divisor) methods also avoid the three fairness paradoxes and can also be used to avoid the "fairness dilemma". Although they can, there are reasons to prefer Webster's method as a method of fair division to both Jefferson's and Adams's method and, in fact, to any other divisor method. In order to sketch those reasons, consider the following table where the percentage-wise satisfaction of the agents' claims in More Horses under Adams's and Jefferson's allocation method is indicated.

Agents	*Claims*	*Adams*	% satisf.	Jefferson	% satisf.
Alice	$c_A = 11$	2	18.2%	1	9.1%
Bob	$c_B = 89$	12	13.5%	14	15.7%
Charlie	$c_C = 5$	1	20%	0	0.0%

Table 2: Satisfaction of claims in More Horses according to Adams and Jefferson.

In the allocation proposed by Adams's method (obtained by taking divisor $d = 8$) for More Horses, Charlie's claim receives $1/5 \cdot 100\% = 20\%$ satisfaction. Hence, Charlie's claim receives more percentage-wise satisfaction than the claims of Alice and Bob. We will also say that *the allocation* $(2, 12, 1)$ *favours Charlie over both Alice and Bob*. Similarly, the allocation $(2, 12, 1)$ favours Alice over Bob since Alice's claim receives more percentage-wise satisfaction than Bob's claim. Note that for any two agents, $(2, 12, 1)$ favours the first agent over the second just in case the first has a smaller claim than the second: we say that $(2, 12, 1)$ favours (agents with) small claims. With respect to the allocation $(1, 14, 0)$ (obtained by taking divisor $d = 6$) that is proposed by Jefferson's method, the situation is completely reversed: for any two agents, $(1, 14, 0)$ favours the first agent over the second, just in case the first has a larger claim than the second. So, the allocation $(1, 14, 0)$ favours (agents with) large claims.

Due to the nature of indivisible goods, any particular allocation of the estate in a (corrected) claims problem will favour some agents over others. This is inescapable and there is nothing wrong with that. However, there is something wrong when an allocation method *systematically* proposes allocations in which agents with small (large) claims are favoured. In other words, there is something wrong with a method that is *biased* towards agents with small (large) claims. The favouritism that the methods of Adams and Jefferson demonstrate in their treatment of More Horses, is exemplary for the biases that these methods demonstrate towards agents with small and large claims, respectively. In contrast, Webster's method does not exhibit such biases, as explained at length in Balinski and Young (2001:76), who conclude that:

> Mathematical analysis shows that of all the divisor methods Webster's is the *only* one that is perfectly unbiased.

The mathematical analysis that Balinski and Young are alluding to consists of various probabilistic models that all propose different ways of making the intuitive notion of "bias" precise. However, all these models point to the same conclusion, which is that Webster's method is the unique unbiased divisor method.

And so, whereas all of the (infinitely many) divisor methods escape the fairness paradoxes, Webster's method is the only one that does so in an unbiased manner. As the ideal of (proportional) fairness dictates that all equally strong claims should receive the same percentage-wise satisfaction, biasedness towards agents with small or large claims is a form of unfairness. We conclude that Webster's method is preferable to all other divisor methods as it is the fairest such method. Any theorist who wants to defend a broadly Broomean theory of fairness and also wants to be congenial to the core feature of Lazenby's account of fairness, must adopt Webster's method as its allocation method.

4.3 Webster's method as a theory of fairness: the road ahead

At this point, let us reflect on the significance of Webster's method for theorising about fairness. We have shown that it can be used as a method for treating the fair division of indivisible goods without having to resort to weighted lotteries, and without succumbing to the three fairness paradoxes. As such, Webster's method seems to be an ideal building block for a fully fledged theory of fairness. It would take us too far to lay out such a fully fledged theory, i.e. a *Webster* *theory of fairness*, in the appropriate amount of detail. Still, there are two objections that we would like to discuss, that may be taken to undermine the prospects for such a Webster theory of fairness.[11]

Firstly, Webster's method may be argued to not treat marginally stronger claims in the right way. For instance, consider a problem where two agents have claims of nearly, but not exactly, equal strength to a single indivisible good. For this problem, Webster's method prescribes that the good should go the agent with the strongest claim whereas fairness requires that the good is allocated via an (almost) equally weighted lottery. Or so *the objection from marginally stronger claims* has it. Lazenby (2014:341-343) discusses this objection that, if successful, can of course also be levelled against his account of fairness. Lazenby provides three independent arguments that purport to establish that the objection from marginally stronger claims is unsuccessful. These arguments can *mutatis mutandis* also be invoked to defend a Webster theory of fairness against the objection from marginally stronger claims. In addition, let us remark that we think there are many contexts in which it is important that stronger claims, even if they are only stronger by a small margin, ought to prevail. Think of claims to seats in parliament, gold medals in competitive sports, prices on the market, and competitive bidding. Those contexts seem very natural to resolve non-probabilistically. As there are quite a few contexts in which it is *intuitively* fair to let a marginally stronger claim win, it seems worthwhile to develop an account of fairness according to which it is also *theoretically* fair to do so. More generally, we concur with Lazenby (2014:342f.) that the validity of the abstract intuition that is echoed in the objection from marginally stronger claims to a large extent depends on how well it conforms to the standards of our everyday ethical practice.

Secondly, we want to consider an objection to the way in which a Webster theory can explain the fairness of *unweighted* (equally weighted) lotteries. When we apply Webster's method to distribute an indivisible good amongst agents with equal claims, the method prescribes that an unweighted lottery must be held. But this is not an *explanation* of the fairness of unweighted lotteries but rather a *stipulation* that such lotteries are fair. On Broome's theory, for instance, the fairness of (un)weighted lotteries is explained in terms of the notion of *surrogate satisfaction*, as briefly discussed in Section 2.1. On an explanation of lottery fairness in terms of surrogate satisfaction, both unweighted and weighted lotteries induce (a "second-best" kind of) fairness. As Webster's method never recommends the use of weighted lotteries, it seems that a Webster theory cannot legitimately appeal to surrogate satisfaction in order to explain the fairness of unweighted lotteries. But this does not prevent a Webster theory to account for this fairness *tout court.* In fact, Wasserman (1996), Stone (2011), and Henning (2015) have all argued that an attempt to defend the fairness of unweighted lotteries in terms of surrogate satisfaction fails. But that does not prevent both Wasserman (1996) and Stone (2011) from providing an alternative type of explanation as to why unweighted lotteries are fair. For instance, Wasserman provides a specific type of procedural account of the fairness of lotteries, that gives substance to the Equal Treatment of Equals principle. As this principle is clearly respected by Webster's method, a Webster theory may account for the fairness of unweighted lotteries via a procedural account.

5 Conclusions

We posed a dilemma for currently existing, broadly Broomean theories of fair division: when faced with indivisible goods, they either (i) must remain silent on a number of important cases or fall prey to Hooker's objection against weighted lotteries, or (ii) come up against three fairness paradoxes. We demonstrated that Webster's method, an allocation method of *apportionment theory*, avoids the dilemma: (i) the method tells us how to allocate the estate in each (corrected) claims problem and it does so without using weighted lotteries and (ii) it does not fall prey to the three fairness paradoxes. Webster's method thus seems to be an ideal building block for a fully fledged theory of fairness.

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Notes

[1]Tomlin (2012) and Kirkpatrick & Eastwood (2015) argue against the core feature of Broome's account, i.e. against the characterisation of fairness as the proportional satisfaction of claims. Hooker (2005) and Saunders (2010) argue that fairness is *not* a strictly comparative value.

[2]Kirkpatrick & Eastwood (2015), Lazenby (2014), Henning (2015), Hooker (2005), Saunders (2010), and Vong (2015) all argue, in one way or another, that Broome misconceives the relation between fairness and weighted lotteries.

[3]Curtis asserts that he does so for reasons given by Hooker (2005) and Saunders (2010).

[4]Curtis claims that method P (and method L) is *implied by* by a more fundamental fairness claim that he calls FC. Effectively this means that Curtis claims that the proportional division rule is *characterised* by FC, i.e. singled out as the only rule satisfying FC. Now, characterising division rules is just what the economic literature on claims problems studies. By drawing on

this literature, we show that Curtis's characterisation of the proportional rule fails, although his attempt to do so raises various interesting questions.

[5]The answer that Curtis's theory does apply to Broomean problems such Medicine as the amount is the only aspect of a claim that is relevant for determining what a fair allocating agent must do (so that strengths can simply be neglected) is clearly unsatisfactory.

[6]Moreover, Broome has only discussed indivisible Broomean problems whose estate consist of a *single* good. For these problems, Broome prescribes to have a weighted lottery in which each agent's chance is in proportion to the strength of his claim and Curtis's Broomean method prescribes exactly the same. This provides further evidence for our claim that, also when the indivisible case is taken into account, our incorporation of claim strengths into Curtis's theory is a natural one.

[7]It should be noted that although Piller thinks that Hooker's objection is a serious one, Piller (2016) also argues that Broome's theory does not fall prey to it. When we apply Broome's theory in order to determine how to fairly allocate the medicine we should, according to Piller (2016:18), answer the question as to "what is more unfair: reducing a small chance to 0 or risking that a much weaker claim is satisfied?" With respect to the answer to this question, Piller (2016:18) remarks that:

> Whatever the answer is one will give to this question, for the purpose of assessing Broome's theory, it is important to realise that this answer comes from outside Broome's theory.

We will not enter the discussion as to whether Piller or other commentators on Broome such as Hooker have interpreted Broome's theory of fairness correctly. We do not need to, for the first horn of our dilemma for theories of fairness merely asserts that theories that use weighted lotteries either fall prey to Hooker's objection or have to remain silent on a number of important cases of fair division. If Piller is right, Broome's theory has to remain silent on cases such as Medicine. This silence then motivates the development of a theory of fairness that, in contrast to Broome's theory, does *not* have to remain silent on cases like Medicine.

[8]Allocating nothing is the most drastic case of "levelling down". Here is a less drastic example. Suppose Ann and Bob have (equally strong) claims to 4 and 8 horses, respectively, whereas there are only 5 horses left. By allocating 1 horse to Ann and 2 horses to Bob their equally strong claims receive equal satisfaction (of 25% each). Thus, by levelling down to 3 horses, a strictly comparative account of fairness can achieve "perfect fairness", whereas such cannot be obtained by allocating all 5 horses. This example of levelling down to 3 horses may be intuitively appealing from the perspective of a strictly comparative account

of fairness. However, according to such an account, levelling down to 0 horses also achieves perfect fairness. A strictly comparative account does not seem to have the means to distinguish "levelling down to 0" from other, perhaps more apt recommendations for how to level down, *in terms of fairness*. Further, suppose that Ann and Bob have (equally strong) claims to 4 and 1 horses respectively whereas there are only 4 horses left. We may allocate 4,3 or 2 horses or just a single horse in any (integer) allocation that we like, but in no such allocation do the equally strong claims of Ann and Bob receive equal satisfaction. Of course, when we allot 0 horses their claims do receive equal satisfaction but suppose that doing so is inappropriate in the context at hand. What allocation(s) then, is (are) prescribed by Lazenby's strictly comparative account when equal satisfaction of claims is unattainable by levelling down? Although several solutions to this problem may come to mind, a proper assessment of the tenability of such solutions requires technical and conceptual work that is beyond the scope of this paper.

[9]For an excellent historical overview of these matters, see the first part of Balinski and Young (2001).

[10]Note that different divisors may induce the same vector of (Webster) rounded d-proportional shares and that the set of all (Webster) feasible allocations for $\mathcal{C}$ is unique, as revealed by (1).

[11]We are grateful to an anonymous referee for raising these questions.

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