From Closure Games to Strong Kleene Truth

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Abstract In this paper, we study the method of closure games, a game-theoretic valuation method for languages of self-referential truth developed by the author. We prove two theorems which jointly establish that the method of closure games characterizes all 3- and 4-valued Strong Kleene fixed points in a novel and informative manner. Amongst others, we also present closure games which induce the minimal and maximal intrinsic fixed point of the Strong Kleene schema.

1 Introduction

Take a language of self-referential truth $L_T$, i.e. a first-order language $L$ with a truth predicate symbol $T$ added to it and in which Liar sentences and their ilk are expressible. Which sentences of $L_T$ are assertible? Which are deniable? These questions are answered by a theory of truth, by which we mean...

... a theory that purports to explain for a first-order language $L_T$ what sentences are assertible [and deniable] in a [ground] model $M$.

The ground model $M$ that Gupta is alluding to is a classical model for $L$, the truth-free fragment of $L_T$. A ground model $M$ equips the sentences of $L$ with a classical valuation $\mathcal{G}_M : Sen(L) \to \{a, d\}$ which determines which sentences of $L$ are assertible and deniable and which is defined as usual[1]. A theory of truth extends $\mathcal{G}_M$ to a valuation of $L_T$ and exploits the extended valuation to specify which sentences of $L_T$ are assertible / deniable in $M$.

In his seminal paper on truth, Kripke [4] specified an inductive method which allows one to define 3-valued fixed point theories of truth. The valuations associated with these theories, called 3-valued fixed points, satisfy the identity of truth:

For all $\sigma \in Sen(L_T)$: $V_M(T(t)) = V_M(\sigma)$, whenever $t$ denotes $\sigma$ in $M$.

Satisfying the identity of truth formally ensures that

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we are entitled to assert (or deny) of any sentence that it is true precisely under the circumstances when we can assert (or deny) the sentence itself. Kripke [4, p701]

Kripke showed how to define 3-valued fixed points for various 3-valued valuation schemas, which differ in the way in which they evaluate logically complex sentences. The fixed points associated with the Strong Kleene schema, 3-valued SK fixed points, have arguably attracted the most attention in the literature on truth. For both philosophical and technical reasons, Visser [6] and Woodruff [11] have generalized Kripke’s 3-valued SK fixed points to 4-valued ones. A 4-valued SK fixed point satisfies the identity of truth and evaluates logically complex sentences according to the Dunn-Belnap (or extended Strong Kleene) schema, which naturally generalizes the Strong Kleene schema to 4-valued logic.

In this paper, we present the method of closure games which allows us to construct 3- and 4-valued SK fixed points in novel and informative manner. As suggested by its name, the method of closure games is a game-theoretic framework in which 3- and 4-valued SK fixed points can be characterized and studied. A game-theoretic approach to theories of truth is not completely novel. For instance, Martin [5] has shown how the minimal (3-valued) SK fixed point over a ground model \(M\), which we denote as \(\mathcal{K}_M\), can be obtained as the outcome of a game, meaning a 2 player perfect information game\(^2\). Martin shows:

**Martin’s game for \(\mathcal{K}_M\).** There is a game \(G_\sigma\) for the minimal SK fixed point \(\mathcal{K}_M\) over a ground model \(M\) so that Player I has a winning strategy in \(G_\sigma\) if and only if \(\mathcal{K}_M(\sigma) = a\); Player II has a winning strategy if and only if \(\mathcal{K}_M(\sigma) = d\); if neither player has a winning strategy, then \(\mathcal{K}_M(\sigma) \notin \{a, d\}\) and the game can be declared a draw.

Welch [7] extends Martin’s work and shows how games can be used to induce the minimal fixed point associated with the Supervaluation schema and also, how certain Herzbergian style revision sequences can be characterized via such games. However, a systematic study of the relation between 2 player perfect information games and (the class of all) 3- and 4-valued SK fixed points is lacking in the literature. In this paper, we will fill this lacuna.

Our closure games differ in some important aspects from the game considered by Martin. Martin associates a single game \(G_\sigma\) with each sentence \(\sigma\) of \(L_T\). Just as in Hintikka’s game-theoretic semantics (cf. [3]), the players in Martin’s game\(^2\) fulfill, at each stage of the game, either the role of verifier or of falsifier. The game \(G_\sigma\) starts with the sentence \(\sigma\) and player I in the role of verifier (and so with II as falsifier) and, as the game proceeds, the players list sentences of \(L_T\) in accordance with rules that reflect the Strong Kleene schema. Whose turn it is does not only depend on the last sentence listed but also on who is, at that stage, the verifier (falsifier). For instance, if the last sentence listed is \(\gamma\), Martin’s rules specify amongst others that

- If \(\gamma\) is \(\alpha \lor \beta\), then the verifier must list \(\alpha\) or list \(\beta\).
- If \(\gamma\) is \(\alpha \land \beta\), then the falsifier must list \(\alpha\) or list \(\beta\).
- If \(\gamma\) is \(T(t)\) and \(t\) denotes a sentence \(\alpha\) of \(L_T\), then the falsifier must list \(\alpha\).
Role changes (between verifier and falsifier) only occur when negated sentences are listed:

- If $\gamma$ is $\neg \alpha$ then the falsifier must list $\alpha$ and the player change roles.

The roles of the players do not only play an important role in the rules of Martin’s game, but also in its winning conditions:

**Winning conditions of Martin’s game** If $\gamma$ is an atomic sentence of $L$ then play terminates; the player who is verifier wins if $\mathcal{C}_M(\gamma) = a$ and loses if $\mathcal{C}_M(\gamma) = d$. The verifier also loses (and play terminates) if $\gamma$ is $T(t)$ with $t$ denoting a non-sentence. If play never terminates, then the game is a draw.

In sharp contrast, the rules and winning conditions of closure games do not involve player roles. This gain in simplicity is achieved by associating two games with each sentence $\sigma$ of $L_T$: the game $G_{A\sigma}$, where player $I$ tries to show that $\sigma$ is assertible and the game $G_{D\sigma}$, where player $I$ tries to show that $\sigma$ is deniable. Before we describe how the closure games $G_{A\sigma}$ and $G_{D\sigma}$ jointly induce a valuation of $\sigma$, we first describe the rules, strategies and winning conditions of our closure games.

**Rules.** The game $G_{A\sigma}$ ($G_{D\sigma}$) starts with the signed sentence $A\sigma$ ($D\sigma$) and, as a closure game proceeds, the players produce a list of signed (with Assertible or Deniable) sentences of $L_T$. Whose turn it is only depends on the last (signed) sentence $X\gamma$ that is listed: $X\gamma$ is either controlled by player $I$ or by player II. The moves that are available to the player who controls $X\gamma$ are described by (assertoric) rules that include:

- If $X\gamma$ is $A\alpha \lor \beta$, then player $I$ must list $A\alpha$ or list $A\beta$.
- If $X\gamma$ is $D\alpha \lor \beta$, then player $II$ must list $D\alpha$ or list $D\beta$.
- If $X\gamma$ is $A\alpha \land \beta$, then player $II$ must list $A\alpha$ or list $A\beta$.
- If $X\gamma$ is $D\alpha \land \beta$, then player $I$ must list $D\alpha$ or list $D\beta$.
- If $X\gamma$ is $A_T(t)$ and $t$ denotes a sentence $\alpha$ of $L_T$, then player $II$ must list $\alpha$.
- If $X\gamma$ is $D_T(t)$ and $t$ denotes a sentence $\alpha$ of $L_T$, then player $I$ must list $\alpha$.

Negated sentences are not associated with role changes but rather with sign changes:

- If $X\gamma$ is $A\neg \alpha$ then player $II$ must list $D\alpha$.
- If $X\gamma$ is $D\neg \alpha$ then player $I$ must list $A\alpha$.

Although our signs can be interpreted as implicitly encoding the two player roles of Martin’s game, we feel that doing so is only confusing. More importantly, our (assertoric) rules have a clear rationale of their own. For instance, when one (i.e. player $I$) wants to show that a disjunction is assertible, one must be able to show that one of the disjuncts (up to one’s choice) is assertible. When one (i.e. player $I$) wants to show that a disjunction is deniable one must be able to show that both disjuncts are deniable; so no matter which disjunct is picked by player $II$, player $I$ must be able to show that it is deniable. The other assertoric rules receive a similar justification.

**Strategies.** Another difference with Martin’s game concerns the notion of a
strategy. In Martin’s game, a strategy of a player is a function from the class of all histories of the game to the set of sentences of \( L_T \), where a history is any finite sequence of sentence-role pairs that can be generated via the game rules. A strategy is thus a quite complicated object. However, it is not hard to show that for Martin’s game, a player has a winning strategy just in case he has a memoryless strategy, i.e. a strategy that does not depend on the history of the game, but only on the last sentence-role pair of the game. The method of closure games restricts itself from the outset to memoryless strategies. That is, a strategy for a player of a closure game is a function from the set of signed sentences that are in his control to the set of signed sentences. For instance, a strategy of player \( \text{I} \) maps \( A_{\alpha \lor \beta} \) to \( A_{\alpha} \) or \( A_{\beta} \) and a strategy of player \( \text{II} \) maps \( D_{\alpha \lor \beta} \) to \( D_{\alpha} \) or \( D_{\beta} \). Strategies in the method of closure games are thus, per definition, quite simple objects. The results of this paper testify that (with respect to characterizing \( \text{SK} \) fixed points) nothing is lost by our restriction to memoryless strategies.

Winning conditions. When, in a closure game \( G_{X_{\sigma}} \), players \( \text{I} \) and \( \text{II} \) pick their strategies, they realize an expansion of \( X_{\sigma} \), a sequence of signed sentences with \( X_{\sigma} \) as its first element and with a successor relation that is determined by the strategies of the players. Expansions are classified into those that result in player \( \text{I} \) winning the game in a ground model \( M \) (these expansions are called open in \( M \)) and into those that do not (these expansions are called closed in \( M \)). Such a bipartition of the set of all expansions into those that are open and closed we call a closure condition. Intuitively, a closure condition may be thought of as an assertoric norm: closed (open) expansions contain assertoric actions that are forbidden (allowed). Player \( \text{I} \) has a winning strategy in the closure game \( G_{X_{\sigma}} \), that is played under closure condition \( \dagger \) in a ground model \( M \) just in case he can ensure that an expansion of \( X_{\sigma} \) is realized that is open in \( M \) according to \( \dagger \). That is, a strategy \( f \) for player \( \text{I} \) is winning just in case, no matter which strategy \( g \) is picked by player \( \text{II} \), the expansion of \( X_{\sigma} \) that is realized by \( f \) and \( g \) is open in \( M \) according to \( \dagger \).

Inducing \( L_T \) valuations by closure games. We say that sentence \( \sigma \) is assertible (deniable) in ground model \( M \) according to closure condition \( \dagger \) just in case player \( \text{I} \) has a winning strategy for \( G_{A_{\sigma}} \) (\( G_{D_{\sigma}} \)) that is played in \( M \) under \( \dagger \). So, relative to a ground model \( M \) and closure condition \( \dagger \), the method of closure game induces a valuation function \( \tau^\dagger_M \) that evaluates \( L_T \) sentences as assertible only, both assertible and deniable, neither assertible nor deniable or as deniable only. Depending on \( \dagger \) and \( M \), \( \tau^\dagger_M \) is either a 2-, 3- or 4-valued function with a range that is a subset of \( \{a, b, n, d\} \). Of course, not all closure conditions will induce \( \text{SK} \) fixed points. One of the main results of this paper is a characterization of those closure conditions that do. In some more detail, the main results of this paper are (organized) as follows.

Structure of the paper
Section 2 presents some general preliminaries.

Section 3 starts with a rigorous presentation of the method of closure games. Then, in Section 3.3, we present two conditions—the world respecting constraint \( \text{WRC} \) and the stable judgement constraint \( \text{SJC} \)—and show that whenever closure condition satisfy our conditions, they induce a \( \text{SK} \) fixed point (cf. Theorem 3.5 and Corollary 3.6).

In Section 3.4, we define some intuitively appealing closure conditions that satisfy \( \text{WRC} \) and \( \text{SJC} \) and study the (3- and 4-valued) \( \text{SK} \) fixed points that they induce. In
particular, we will present closure conditions that induce two versions of the minimal fixed point, having range \{a, n, d\} and \{a, b, d\} respectively.

In Section 3.5 we show, conversely, that any SK fixed point can be induced from closure conditions that satisfy WRC and SJC (cf. Theorem 3.14 and Corollary 3.15). To do so, we take a SK fixed point \(V_M\) and define closure conditions that satisfy WRC and SJC in terms of \(V_M\). To induce a SK fixed point in this way is, in some sense, "cheating", as we put \(V_M\) in to get \(V_M\) out". In contrast, the closure conditions presented in Section 3.4 are clear examples of "non-cheating closure conditions".

Section 4 will be, amongst others, devoted to finding "non-cheating closure conditions" for the maximal intrinsic SK fixed point (cf. [4]).

In section 4.1, we define a slight modification of the method of closure games, which we call assertoric semantics. Whereas the method of closure games induces \(L_T\) valuations by putting closure conditions on sequences of signed sentences (expansions), assertoric semantics does so by putting closure conditions on sets of signed sentences.

In section 4.2, we show how assertoric semantics induces the minimal SK fixed point (with range \{a, b, d\}) and also, how it induces Kripke’s 4-valued “modal theory of truth” \(\mathcal{K}^4\), which he defined (implicitly) in [4] by quantifying over all 3-valued SK fixed points; for instance, according to \(\mathcal{K}^4\) the Liar is paradoxical as there is no 3-valued SK fixed point in which it is evaluated as \(a\) and also, there is no 3-valued SK fixed point in which it is evaluated as \(d\).

In section 4.3 we show how our characterization of \(\mathcal{K}^4\) (via assertoric semantics) allows us to define “non-cheating closure conditions” that induce (via the method of closure games) two versions of the maximal intrinsic SK fixed point, having range \{a, n, d\} and \{a, b, d\} respectively.

Section 5 concludes.

2 Preliminaries

\(L_T\) will denote a first order language with identity (\(\approx\)), a truth predicate (\(T\)) and with a quotational name ([\(\sigma\)]) for each sentence \(\sigma\) of \(L_T\). \(L\) will denote the language that is exactly like \(L_T\), except for the fact that it does not contain the truth predicate \(T\). A ground model \(M = (D, I)\) is a classical model for of \(L\) such that \(Sen(L_T) \subseteq D\) and such that \(I([\sigma]) = \sigma\) for all \(\sigma \in Sen(L_T)\). A sentence may be denoted in various ways; \(\sigma\) will be used to denote any closed term, quotational name or not, that denotes \(\sigma\) in \(M\). For each ground model \(M = (D, I)\), we will (tacitly) expand our language \(L_T\) to a language \(L_T + M\) which has, in addition to the vocabulary of \(L_T\), constant symbols available to refer to all the members of the domain of \(M\). This (tacit, we will always simply speak of \(L_T\)) expansion has the advantage that quantification can be treated substitutionally, so that we do not need to be bothered with variable assignments.

Observe that a ground model may, but need not, define self-referential sentences such as the Liar, i.e. a sentence that says, of itself that it is not true. It will turn out convenient to fix some notation pertaining to some canonical self-referential sentence such as the Liar and the Truthsteller.

**Definition 2.1 Some notational conventions**

In this paper, the non-quotational constants \(\lambda, \tau, \eta, \theta\) and \(\mu\) will be used as follows, where \(I\) is some interpretation function.

1. \(I(\lambda) = \neg T(\lambda)\). We say that \(\neg T(\lambda)\) is a Liar.
2. \( I(\tau) = T(\tau) \). We say that \( T(\tau) \) is a Truthteller.
3. \( I(\eta) = T(\eta) \lor \neg T(\eta) \). We say that \( T(\eta) \lor \neg T(\eta) \) is a Tautologyteller.
4. \( I(\theta) = T(\theta) \land \neg T(\theta) \). We say that \( T(\theta) \land \neg T(\theta) \) is a Contradictionteller.
5. \( I(\mu) = T(c_0) \) where, for each \( n \), \( I(c_n) = \neg T(c_{n+1}) \). We say that \( T(\mu) \) is an Unstabilityteller.

To be sure, the notational convention does not imply that every ground model contains a Liar: given an interpretation function \( I \) a constant \( \lambda \) satisfying \( I(\lambda) = \neg T(\lambda) \) may not exist. Similar remarks apply to the Truthteller, Tautologyteller, Contradictionteller and Unstabilityteller.

Given a ground model \( M \), \( \mathcal{G}_M : \text{Sen}(L) \rightarrow \{a,d\} \) denotes the classical valuation of \( L \) based on \( M \) and is defined as usual. A theory of truth \( T \) takes a ground model \( M \) as input and outputs a valuation \( T_M \) of the sentences of \( L_T \). That is, \( T \) outputs a function \( T_M : \text{Sen}(L_T) \rightarrow V \), where \( V \) contains the (semantic) values recognized by \( T \). We assume, without loss of generality, that when \( T \) is a theory of truth, \( a \) and \( d \) are always amongst the semantic values recognized by \( T \). Not any semantic valuation of the sentences of \( L_T \) qualifies as the valuation of a theory of truth. In this paper, we assume that in order for \( T \) to qualify as a theory of truth, \( T_M \) should respect the world and the identity of truth, as defined below. Besides these two familiar conditions we impose one further, arbitrary but technically convenient, condition: every truth ascription to an object that is not a sentence is to be evaluated as \( d \) by a theory of truth \( T \).

**Definition 2.2 Theory of truth**

Let \( T \) be a valuation method which, given a ground model \( M = (D,I) \), outputs a valuation function \( T_M : \text{Sen}(L_T) \rightarrow V \). We say that \( T \) is a theory of truth just in case, for every ground model \( M \), we have

\[
\forall \sigma \in \text{Sen}(L) : \mathcal{G}_M(\sigma) = T_M(\sigma) \tag{1}
\]

\[
\forall \sigma \in \text{Sen}(L_T) : T_M(T(\overline{\sigma})) = T_M(\sigma) \tag{2}
\]

\[
T_M(T(i)) = d \text{ whenever } I(i) \notin \text{Sen}(L_T) \tag{3}
\]

That is, \( T_M \) should (1) respect the world and (2) the identity of truth, while (3) all truth ascriptions to non-sentences are evaluated as \( d \).

We will be particularly interested in theories of truth that output Strong Kleene (SK) valuations. We distinguish between 2-valued, 3-valued and 4-valued SK valuations, where a 2-valued SK valuation is just a classical valuation. In a 3-valued SK valuation, logically complex sentences are evaluated according to the Strong Kleene schema and in a 4-valued SK valuation, these are evaluated according to the extended Strong Kleene (or Dunn-Belnap) schema. It will be convenient to distinguish between two types of 3-valued SK valuations: those with range \( \{a,n,d\} \) and those with range \( \{a,b,d\} \). SK valuations of the first type we call \( 3n \)-valued SK valuations, of the second type \( 3b \)-valued SK valuations.

**Definition 2.3 Strong Kleene valuations**

Let \( V_M : \text{Sen}(L_T) \rightarrow V \) be a valuation of \( L_T \) in \( M \) such that \( V \) is either \( \{a,d\} \), \( \{a,n,d\} \), \( \{a,b,d\} \) or \( \{a,b,n,d\} \). We say that \( V_M \) is a 2-valued (\( 3n \)-valued, \( 3b \)-valued, 4-valued) SK valuation just in case, with \( V \) the lattice associated with its range as in Figure 1, we have

- \( \neg \) swaps \( a \) for \( d \) and vice versa and leaves other values unchanged.
- $\land$ and $\lor$ act, respectively as meet and join on $V_\leq$.
- $\forall$ and $\exists$ act, respectively, as generalized meet and join on $V_\leq$.

Observe that the notion of a Strong Kleene valuation does not mention the semantic behavior of the truth predicate, nor the relation with the valuation of $L$ as induced by the ground model $M$. It will turn out to be convenient to separate the notion of a Strong Kleene valuation from the notion of an SK fixed point, by which we mean a Strong Kleene valuation that respects the defining clauses of a theory of truth.

**Definition 2.4**  
**SK fixed points, FP$^a(M)$ and FP$^b(M)$**  
Let $V_M : Sen(L_T) \to V$ be an SK valuation of $L_T$ in $M$. We say that $V_M$ is an SK fixed point over $M$ just in case $V_M$ satisfies clauses (1), (2) and (3) of Definition 2.2. We will use $FP^a(M)$ to denote the set of all 2- and 3$^a$-valued SK fixed points over $M$, whereas $FP^b(M)$ will denote the set of all 2- and 3$^b$-valued SK fixed points over $M$.

A Strong Kleene theory of truth (SK theory) is a theory of truth that assigns an SK fixed point to each ground model $M$.

**Definition 2.5**  
**SK theory of truth**  
Let $T$ be a theory of truth. We say that $T$ is an SK theory just in case, for every ground model $M$, $T_M$ is an SK fixed point. An SK theory that recognizes 3 (4) semantic values is called an SK$_3$ theory (SK$_4$ theory).

Note that there are no SK theories that recognize only two semantic values, as is testified by a ground model $M = (D, I)$ that contains a Liar $\neg T(\lambda)$. On the other hand, some ground models allow $L_T$ to be valued by a 2-valued SK fixed point. Also, note that the definition of an SK theory is quite liberal. A “genuine” SK theory $T$ must, arguably, consist of a systematic way in which an arbitrary ground model $M$ is converted into an SK fixed point $T_M$, and the notion of a “systematic conversion” does not appear in our definition. However, the definition as given is just fine for our purposes.

Two interesting and well-known SK$_3$ theories are Kripke’s minimal fixed point theory and his maximal intrinsic fixed point theory. In line with Definition 2.3, we distinguish a 3$^a$-valued and a 3$^b$-valued version of both theories. In order to define the 3$^n$-valued versions of those theories, we define the following partial order on $FP^n(M)$. With $V_M, V_M' \in FP^n(M)$, we let

$$V_M \leq V_M' \iff \forall \sigma \in Sen(L_T) : V_M(\sigma) = a \Rightarrow V_M'(\sigma) = a$$

When $V_M \leq V_M'$ and $V_M \neq V_M'$, we write $V_M < V_M'$. We say that $V_M$ is maximal just in case for no $V_M'$ do we have $V_M < V_M'$. We say that $V_M$ is minimal just in case for no $V_M'$ do we have $V_M' < V_M$. We say that $V_M$ and $V_M'$ are compatible just in case there exists

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**Figure 1** Hasse diagrams of lattices on \{a, d\}, \{a, n, d\}, \{a, b, d\} and \{a, b, n, d\}. 

![Hasse diagrams](image-url)
a $V^*_M \in \text{FP}^n(M)$ which extends them both: $V_M \leq V^*_M$ and $V'_M \leq V^*_M$. $V_M$ is called intrinsic just in case it is compatible with every other fixed point in $\text{FP}^n(M)$. For any ground model $M$, we let $\text{I}^n(M)$ be the set of all intrinsic fixed points over $M$. As [4] observes, $\text{I}^n(M)$ has a maximum element and $\text{FP}^n(M)$ has a least element with respect to the relation $\preceq$. The $3n$-valued minimal fixed point $\mathcal{K}$ and the $3n$-valued maximal intrinsic fixed point $\mathcal{K}^+$ can now be defined as follows.

**Definition 2.6** $\mathcal{K}$ and $\mathcal{K}^+$

Let $M$ be a ground model. According to the theory $\mathcal{K}$, the valuation of $L_T$ in $M$ is given by $\mathcal{K}_M : \text{Sen}(L_T) \rightarrow \{a, n, d\}$, where $\mathcal{K}_M$ is the minimum of $\text{FP}^n(M)$. According to the theory $\mathcal{K}^+$, the valuation of $L_T$ in $M$ is given by $\mathcal{K}^+_M : \text{Sen}(L_T) \rightarrow \{a, n, d\}$, where $\mathcal{K}^+_M$ is the maximum of $\text{I}^n(M)$.

The $3b$-valued versions of the minimal and maximal intrinsic fixed point will be denoted as $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^+$ respectively. For sake of definiteness, their definitions are as follows.

**Definition 2.7** $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^+$

Let $M$ be a ground model. According to the theory $\overline{\mathcal{K}}$, the valuation of $L_T$ in $M$ is given by $\overline{\mathcal{K}}_M : \text{Sen}(L_T) \rightarrow \{a, b, d\}$, where $\overline{\mathcal{K}}_M(\sigma) = b$ iff $\mathcal{K}_M(\sigma) = n$ and $\overline{\mathcal{K}}_M(\sigma) = \mathcal{K}_M(\sigma)$ otherwise. According to the theory $\overline{\mathcal{K}}^+$, the valuation of $L_T$ in $M$ is given by $\overline{\mathcal{K}}^+_M : \text{Sen}(L_T) \rightarrow \{a, b, d\}$, where $\overline{\mathcal{K}}^+_M(\sigma) = b$ iff $\mathcal{K}^+_M(\sigma) = n$ and $\overline{\mathcal{K}}^+_M(\sigma) = \mathcal{K}^+_M(\sigma)$ otherwise.

### 3 The Method of Closure Games

#### 3.1 Defining the method of closure games

In the introduction we sketched the method of closure games and its central notions such as assertoric rules, strategies, expansions and closure conditions. In this section, we turn the previous sketches into precise definitions.

**Some preliminary notions.** A signed (with $A$ or $D$) sentence of $L_T$ will be called an AD sentence. $\mathcal{X}$ denotes the set of all AD sentences.

$$\mathcal{X} = \{X_\sigma \mid X \in \{A, D\}, \sigma \in \text{Sen}(L_T)\}$$

With $\text{Ar}(L)$, we denote the set of atomic sentences of $L$. These sentences are assumed to receive their (classical) valuation from the ground model $M$ and can be thought of as the “non-semantic facts”. We will treat (atomic) truth ascriptions to non-sentential objects on a par with members of $\text{Ar}(L)$. Hence, it is convenient to define, with $M = (D, I)$, the set $\text{Ar}_M(L)$ as follows:

$$\text{Ar}_M(L) = \text{Ar}(L) \cup \{T(t) \mid I(t) \not\in \text{Sen}(L_T)\}$$

**Assertoric rules.** In a closure game the two players produce a list of AD sentences. Whose turn it is only depends on the last AD sentence $X_T$ that is listed and is determined by assertoric rules that include:

- If $X_T$ is $A_\alpha \land \beta$, then player $I$ must list $A_\alpha$ or list $A_\beta$.
- If $X_T$ is $D_\alpha \land \beta$, then player $II$ must list $D_\alpha$ or list $D_\beta$.

In order to present all assertoric rules in a uniform manner, it is convenient to first introduce some notation. We will say that player $I$ controls $A_\alpha \land \beta$ and that player
II controls $D_{\alpha\lor\beta}$. More generally, each AD sentence $X_\gamma$ is controlled by one of the players. With respect to an AD sentence $X_\gamma$ that is in his control, a player always has to list a single element of the set $\Pi(X_\gamma)$, consisting of all immediate AD subsentences of $X_\gamma$. Thus, $\Pi(A_{\alpha\lor\beta}) = \{A_{\alpha}, A_{\beta}\}$ and $\Pi(D_{\alpha\lor\beta}) = \{D_{\alpha}, D_{\beta}\}$. The general form of an assertoric rule can then be depicted as follows

$$
\frac{X_\gamma}{\Pi(X_\gamma) J} \quad \text{(where } J \in \{I, II\} \text{)}
$$

Thus, (4) states that player $J$ controls $X_\gamma$ and so, with respect to $X_\gamma$, player $J$ has to pick an AD sentence in $\Pi(X_\gamma)$. By exploiting the notation just introduced, Figure 2 below states the assertoric rules.

With respect to Figure 2, observe that (signed) negations, truth ascriptions and elements of $At_\delta(L)$, it does not matter which player, $I$ or $II$ controls them. The actual allotment of player control to those sentences was chosen for sake of symmetry only: if player $I$ controls $A_\sigma$, player $II$ controls $D_\sigma$ and vice versa. The reason that there are also (trivial) assertoric rules pertaining to elements of $At_\delta(L)$ will be explained below, where we define the notion of an expansion.

| \(\neg\) | $A_{\neg\alpha} II$ | $D_{\neg\alpha} I$ |
| \(\lor\) | $A_{\alpha \lor \beta} I$ | $D_{\alpha \lor \beta} II$ |
| \(\land\) | $A_{\alpha \land \beta} I$ | $D_{\alpha \land \beta} II$ |
| \(\exists\) | $\{A_{\exists \theta(i)} \mid t \in CTerm(L_T)\} II$ | $\{D_{\exists \theta(i)} \mid t \in CTerm(L_T)\} II$ |
| \(\forall\) | $\{A_{\forall \theta(i)} \mid t \in CTerm(L_T)\} I$ | $\{D_{\forall \theta(i)} \mid t \in CTerm(L_T)\} I$ |
| $T$ | $A_{\sigma} II$ | $D_{\sigma} I$ |
| $\sigma \in At_\delta(L)$ | $A_{\sigma} II$ | $D_{\sigma} I$ |

Figure 2 The assertoric rules

**Strategies.** A player’s strategy determines the moves that the player will take at any stage of the game. As was announced in the introduction, the method of closure games restricts itself to memoryless strategies. This means that the moves of a player do not depend on the history of the game—i.e., on the list of AD sentences that has been produced thus far—but only on the last element of the list. In other words, a strategy of a player in a closure game is a function that maps each AD sentence $X_\sigma$ that is in his control to an element of $\Pi(X_\sigma)$.

**Definition 3.1 Strategies and strategy sets**

A strategy for player $I$ is a function $f$ that maps each $X_\sigma$ that is controlled by $I$ to an element of $\Pi(X_\sigma)$. The set of all strategies of player $I$ is denoted by $\mathcal{F}_I$.

A strategy for player $II$ is a function $g$ that maps each $X_\sigma$ that is controlled by $II$ to an element of $\Pi(X_\sigma)$. The set of all strategies of player $II$ is denoted by $\mathcal{G}_I$. 

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This page contains technical content from a document related to logic and game theory. The text explains the relationship between AD sentences and the control of sentences by players in a closure game. It introduces the concept of strategies and strategy sets, which are functions that map AD sentences to elements of the set $\Pi(X_\sigma)$. The emphasis is on memoryless strategies, where the moves of a player do not depend on the history of the game. The page also includes a table that lists the assertoric rules in the form of logical expressions. This table is used to define the method of closure games, which restricts strategy choices to immediate AD subsentences of a given sentence. The page concludes with a definition of strategies and strategy sets, distinguishing between strategies for player $I$ and player $II$.
Expansions. Let $X_{\sigma}$ be an AD sentence. A pair of strategies $f$ (for player $I$) and $g$ (for player $II$) realizes an expansion of $X_{\sigma}$, i.e., a sequence of AD sentences that has $X_{\sigma}$ as its first term and with a successor relation that is determined by $f$ and $g$. As an example, consider the AD sentence $A_{P(c_1)\cap P(c_2)}$, where $P(c_1)$ and $P(c_2)$ are atomic sentences of $L$. A strategy $g$ of player $II$ according to which $g(A_{P(c_1)\cap P(c_2)}) = A_{P(c_1)}$ (combined with any strategy $f$ for player $I$) realizes the following expansion of $A_{P(c_1)\cap P(c_2)}$:

$$A_{P(c_1)\cap P(c_2)}, A_{P(c_1)}, A_{P(c_1)}\ldots$$ (5)

Indeed, due to the (trivial) assertoric rules pertaining to elements of $At_M^+(L)$, an expansion is an infinite sequence of AD sentences.

**Definition 3.2** Expansions and the set $\text{EXP}_M$

With $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $X_{\sigma} \in \mathcal{X}$, $\exp(X_{\sigma}, f, g)$ denotes the expansion of $X_{\sigma}$ by $f$ and $g$. The set of all expansions in $M$ is denoted by $\text{EXP}_M$.

**Closure conditions.** A closure game $G_{X_{\sigma}}$ starts with AD sentence $X_{\sigma}$ and, when the players pick their respective strategies $f$ and $g$, results in the expansion $\exp(X_{\sigma}, f, g)$. A closure game is always played in, or relative to, a ground model $M$. In a closure game $G_{X_{\sigma}}$, player $I$ tries to show that $\sigma$ is assertible in $M$ whereas in the game $G_{D_{\sigma}}$, player $I$ tries to show that $\sigma$ is deniable in $M$. Player $I$ succeeds in showing the assertibility (deniability) of $\sigma$ in $M$ just in case he has a winning strategy in $G_{X_{\sigma}}$ ($G_{D_{\sigma}}$), where a strategy $f$ for player $I$ in $G_{X_{\sigma}}$ ($G_{D_{\sigma}}$) is winning just in case $f$ ensures that an expansion of $A_{\sigma}$ ($D_{\sigma}$) is realized that is open in $M$.

Which expansions are open in $M$ (and which are closed, i.e. not open) depends on the closure condition under which a game is played. Intuitively, a closure condition may be thought of as describing an assertoric norm. Formally, a closure condition is defined as follows.

**Definition 3.3** Closure conditions

A closure condition $\dagger = \{O^\dagger_M, C^\dagger_M\}$ is a bipartition of $\text{EXP}_M$ into the sets $O^\dagger_M \neq \emptyset$, consisting of the open, expansions in $M$, and $C^\dagger_M \neq \emptyset$, containing the closed, expansions in $M$.

Let $G_{X_{\sigma}}$ be a closure game played in ground model $M$ under closure condition $\dagger = \{O^\dagger_M, C^\dagger_M\}$. When player $I$ has a winning strategy in $G_{X_{\sigma}}$, we write $O^\dagger_M(X_{\sigma})$:

$$O^\dagger_M(X_{\sigma}) \iff \exists f \in \mathcal{F} \forall g \in \mathcal{G} : \exp(X_{\sigma}, f, g) \in O^\dagger_M$$

We also let

$$C^\dagger_M(X_{\sigma}) \iff \text{not } O^\dagger_M(X_{\sigma})$$

**Inducing $L_T$ valuations.** Given a ground model $M$ and closure condition $\dagger$, the method of closure games induces $\gamma^\dagger_M$, a valuation of $L_T$ in $M$, as follows:

$$\gamma^\dagger_M(\sigma) = \begin{cases} a, & O^\dagger_M(A_{\sigma}) \text{ and } C^\dagger_M(D_{\sigma}); \\ b, & O^\dagger_M(A_{\sigma}) \text{ and } O^\dagger_M(D_{\sigma}); \\ c, & C^\dagger_M(A_{\sigma}) \text{ and } C^\dagger_M(D_{\sigma}); \\ d, & C^\dagger_M(A_{\sigma}) \text{ and } O^\dagger_M(D_{\sigma}). \end{cases}$$
3.2 Defining closure conditions: classifying expansions

The method of closure games induces $L_T$ valuations by closure conditions, that is, by declaring expansions open or closed. This section introduces some classifications of expansions which facilitate the definition of closure conditions later on.

Here are six examples of expansions, which will be used to illustrate our classifications of expansions. The examples involve $P(c)$, an atomic sentence of $L$ and a Liar, Truth-teller and Unstabilityteller as defined in Definition 2.1.

1. $A_{P(c)} \vee \neg P(c), A_{P(c)} \cdot A_{P(c)}$, ...
2. $A_{\neg P(c)}, D_{P(c)}$, ...
3. $A_{P(c)}, A_{T(c)}, A_{\neg T(c)}$, ...
4. $D_{T(c)}, D_{\neg T(c)}$, ...
5. $A_{T(c)}, A_{\neg T(c)}, D_{T(c)}, D_{\neg T(c)}$, ...
6. $A_{\neg T(c)}, A_{T(c)}, D_{T(c)}, D_{\neg T(c)}$, ...

First, observe that every expansion is either stable, stable or unstable. The formal definition of these notions is clear from the remark that expansions 1 and 3 are stable, 2 and 4 are stable and 5 and 6 are unstable. Next, observe that every expansion is either grounded or ungrounded, where an expansion is grounded just in case it contains, for some $\sigma \in At_M(L)$, $x_\sigma$; we say that $x_\sigma$ is the ground of the expansion. Grounded expansions are either correct in $M$ or incorrect in $M$. An expansion is correct in $M$ just in case its ground is contained in the world $w_M$, which is defined as follows

$$w_M = \{ A_{\sigma} \mid E_M(\sigma) = a, \sigma \in At(L) \} \cup \{ D_{\sigma} \mid E_M(\sigma) = d, \sigma \in At(L) \} \cup \{ D_{T_{\alpha}} \mid I(\alpha) \notin Sem(L_T) \}$$

The definition of $w_M$ reveals that, in line with Definition 2.2, we assume that (atomic) sentences that ascribe truth to non-sentential objects always have to be denied. Now, when we assume a ground model $M$ in which $E_M(P(c)) = a$, expansion 1 is grounded and correct (as $A_{P(c)} \in w_M$) while expansion 2 is grounded and incorrect (as $A_{P(c)} \notin w_M$). Expansions 3, 4, 5 and 6 are ungrounded. An (ungrounded) expansion is vicious just in case it contains a vicious cycle, or in other words, an expansion $\{y_n\}_{n \in \mathbb{N}}$ is vicious just in case

$$\exists \sigma \forall n \exists m, m' > n : y_m = A_\sigma \text{ and } y_{m'} = D_\sigma$$

Indeed, expansion 6 is vicious. We now introduce the following abbreviations for subsets of $\text{EXP}_M$.

Definition 3.4 Classifying expansions

We define the following subsets of $\text{EXP}_M$.

- $G_M$: the set of all grounded expansions.
- $U_M$: the set of all ungrounded expansions.
- $G_{\text{cor}}$: the set of all grounded and correct expansions.
- $G_{\text{inc}}$: the set of all grounded and incorrect expansions.
- $U_{\text{vict}}$: the set of all (ungrounded) vicious expansions.
- $U_{\text{inc}}$: the set of all ungrounded non-vicious expansions.
- $U_{\text{st}}$: the set of all ungrounded stable $A$ expansions.
- $U_{\text{st}}$: the set of all ungrounded stable $D$ expansions.
- $U_{\text{un}}$: the set of all (ungrounded) unstable expansions.
### 3.3 The first stable judgement theorem

In this section, we prove our first stable judgement theorem which states that if closure conditions satisfy what we call the stable judgement constraint, they induce an SK valuation that respects the identity of truth. We further show that if, in addition, closure conditions satisfy the world respecting constraint, they induce an SK fixed point.

For any expansion $\exp$, let $\exp'$ denote the successor expansion of $\exp$, by which we mean the expansion that is obtained by removing the first term of $\exp$. A closure condition $\dagger = \{O_M^1, C_M^1\}$ satisfies the stable judgement constraint (SJC), just in case, for every expansion $\exp \in \text{EXP}_M$, we have

\[ \text{SJC} : \exp \in C_M^1 \iff \exp' \in C_M^1 \]

Note that, equivalently, SJC can be formulated in terms of openness:

\[ \text{SJC} : \exp \in O_M^1 \iff \exp' \in O_M^1 \]

Below, we prove the first stable judgement theorem, in which we refer to the set of all AD subsentences of $X_\sigma$, denoted $\Pi(X_\sigma)$. Formally, $\Pi(X_\sigma)$ is defined by taking the transitive closure of the binary relation induced by the set of all immediate AD subsentences of $X_\sigma$:

- $\Pi(\cdot, \cdot)$ is defined by: $\Pi(X_\sigma, Y_\alpha) \Rightarrow Y_\alpha \in \Pi(X_\sigma)$.
- $\Pi(\cdot, \cdot)$ is defined as the transitive closure of $\Pi(\cdot, \cdot)$.
- $\Pi(\cdot)$ is defined by: $\Pi(X_\sigma) = \{ Y_\alpha \mid \Pi(X_\sigma, Y_\alpha) \}$

#### Theorem 3.5 First stable judgement theorem

Let $M = (D,I)$ be a ground model, let $\dagger = \{O_M^1, C_M^1\}$ be a closure condition that satisfies SJC and let $\dagger'$ be the valuation function induced by $\dagger$. We have

1. $\dagger'$ is either a 2-, 3n-, 3b or 4-valued Strong Kleene valuation (see Definition 2.3).
2. For each $\sigma \in \text{Sen}(L_T)$ we have $\dagger'(T(\sigma)) = \dagger'(\sigma)$, That is $\dagger'$ respects the identity of truth.

#### Proof

Let $\dagger = \{O_M^1, C_M^1\}$ be a closure condition that satisfies SJC. Notice that, in order to show that $\dagger'$ is a Strong Kleene valuation that respects the identity of truth, it suffices to show that for every AD sentence $X_\sigma$:

- player I controls $X_\sigma \Rightarrow \left( O_M^1(X_\sigma) \iff \exists Y_\alpha \in \Pi(X_\sigma) : O_M^1(Y_\alpha) \right)$
- player II controls $X_\sigma \Rightarrow \left( O_M^1(X_\sigma) \iff \forall Y_\alpha \in \Pi(X_\sigma) : O_M^1(Y_\alpha) \right)$

We illustrate for $A_{\alpha \land \beta}$. Other cases are similar and left to the reader.

$\Rightarrow$ Suppose that $O_M^1(A_{\alpha \land \beta})$. This means that there is a strategy $f \in \mathcal{F}$ such that for all $g \in \mathcal{G}$, $\exp(A_{\alpha \land \beta}, f, g)$ is open. Now $A_{\alpha \land \beta}$ is controlled by player II, and the strategies of player II can be bi-partitioned into strategies of type $g_\alpha$, which have $g(A_{\alpha \land \beta}) = A_\alpha$ and strategies of type $g_\beta$, which have $g(A_{\alpha \land \beta}) = A_\beta$. As $f$ results in an open expansion, no matter whether player II plays a strategy of type $g_\alpha$ or $g_\beta$, it follows, as $\dagger$ satisfies SJC, that $f$ is such that for all $g \in \mathcal{G}$, we have $\exp(A_{\alpha}, f, g) \in O_M^1$ and that $\exp(A_\beta, f, g) \in O_M^1$. Hence, $O_M^1(A_\alpha)$ and $O_M^1(A_\beta)$.

$\Leftarrow$ Suppose that $O_M^1(A_\alpha)$ and $O_M^1(A_\beta)$. This means that there exists a strategy $f_\alpha \in \mathcal{F}$ such that for all $g \in \mathcal{G}$ we have $\exp(A_{\alpha}, f_\alpha, g) \in O_M^1$ and that there exists a
strategy $f_\beta \in \mathcal{F}$ such that for all $g \in \mathcal{G}$ we have $\exp(A_\beta, f_\beta, g) \in O_M^\dagger$. Let $f \in \mathcal{F}$ be any strategy that satisfies:

- $X_\sigma \in \Pi(A_\alpha) \cdot$ player I controls $X_\sigma \Rightarrow f(X_\sigma) = f_\alpha(X_\sigma)$
- $X_\sigma \in (\Pi(A_\beta) - \Pi(A_\alpha)) \cdot$ player I controls $X_\sigma \Rightarrow f(X_\sigma) = f_\beta(X_\sigma)$

From the fact that $\dagger$ satisfies SJC, it follows that the constructed $f$ is such that for all $g \in \mathcal{G}$ we have $\exp(A_\alpha \land \beta, f, g) \in O_M^\dagger$.

Thus, picking a closure condition that satisfies SJC ensures that we induce a Strong Kleene valuation that respects the identity of truth. As such, a closure condition that satisfies SJC does not guarantee that we induce an $SK$ fixed point (as defined by Definition 2.4). However, by posing the following additional constraint on closure conditions, we ensure that they induce fixed point valuations. Let $\dagger = \{ O_M^{cor}, C_M^{inc} \}$ be any closure condition. We say that $\dagger$ satisfies the world respecting constraint, WRC, just in case

$$WRC: G\text{cor}_M \subseteq O_M^\dagger \text{ and } G\text{inc}_M \subseteq C_M^\dagger$$

We get the following corollary to Theorem 3.5.

**Corollary 3.6** First stable judgement corollary

Let $\dagger = \{ O_M^{cor}, C_M^{inc} \}$ be a closure condition that satisfies WRC and SJC. Then $\forall_M^\dagger$ is an (2-, 3- or 4-valued) $SK$ fixed point over $M$ (cf. Definition 2.4).

**Proof** In light of Theorem 3.5, it suffices to show that if $\dagger$ satisfies WRC, then $\forall_M^\dagger$ satisfies clauses (1) and (3) of Definition 2.2. A proof can be given by induction on the complexity of sentences of $L$, accounting for the non-sentential truth ascriptions in a straightforward way.

### 3.4 Putting the first stable judgement theorem to work

In this section, we put the first stable judgement theorem to work; we define closure conditions that satisfy SJC and WRC and that, accordingly, induce $SK$ fixed points. Amongst others, we present closure conditions that induce the $3n$- and $3b$-valued version of the minimal $SK$ fixed point.

Consider the following closure conditions.

- **gr(oundedness) closure conditions**: $O_M^{gr} = G_M^{cor}$
- ** Closure conditions**: $O_M^{\bullet} = G_M^{cor} \cup U_M^{inc}$

It is easily seen that those closure conditions satisfy SJC and WRC. Hence, by the first stable judgement theorem, $\forall^{gr}$ and $\forall^{\bullet}$ are $SK$ theories of truth. In fact, we have

**Proposition 3.7** $\forall^{gr} = \mathcal{X}$ whereas $\forall^{\bullet}$ is an $SK_4$ theory.

**Proof** In [8] we showed that $\forall^{gr} = \mathcal{X}$. The fact that $\forall^{\bullet}$ is an $SK_4$ theory follows from the observation (that the $\bullet$ closure conditions satisfy SJC and WRC and) that $\forall_M^{\bullet}(-T(\lambda)) = n$ while $\forall_M^{\bullet}(T(\tau)) = b$.

It is instructive to explain, in terms of the method of closure games, why $\forall_M^{gr}$ is 3-valued, whereas $\forall_M^{\bullet}$ is 4-valued\textsuperscript{11}. In order to do so, we will prove a useful lemma which requires the following definition of the inverses of $AD$ sentences and (sets of) expansions:
Definition 3.8  **Inverses**  
For each AD sentence $X_\sigma$, we define its *inverse* $X_\sigma^{-1}$ by stipulating that $A_\sigma^{-1} = D_\sigma$ and $D_\sigma^{-1} = A_\sigma$.

For each expansion $\exp = \{y_n\}_{n \in \mathbb{N}}$, we define its *inverse expansion* $\exp^{-1} = \{z_n\}_{n \in \mathbb{N}}$ by letting, for any $n \in \mathbb{N}$,

$$z_n = X_\sigma \iff y_n = X_\sigma^{-1}$$

We define the *inverse* $S^{-1}$ of a set of expansions $S$ by stipulating that $S^{-1} = \{\exp^{-1} | \exp \in S\}$.

Here is the announced lemma.

**Lemma 3.9**  
$\exists f \forall g \ \exp(X_\sigma, f, g) \in S \iff \exists g \forall f \ \exp(X_\sigma^{-1}, f, g) \in S^{-1}$.

**Proof**  
For each strategy $f$ of player I, there is a *mirror strategy* for player II, call it $g_f$, that is defined as follows:

$$g_f(X_\alpha) = Y_\beta \iff f(X_\alpha^{-1}) = Y_\beta^{-1}$$

Similarly, for each strategy $g$ of player II, there is a mirror strategy for player I which may be called $f_g$. The lemma readily follows from an inspection of the notion of a mirror strategy. $\square$

In order to explain the 3-valuedness of of $\Upsilon_M^{gr}$ in terms of the method of closure games, observe that, as the set of expansions $G_M^{cor}$ is the inverse of $G_M^{inc}$, it follows from Lemma 3.9 that

$$\exists f \forall g \ \exp(X_\sigma, f, g) \in G_M^{cor} \iff \exists g \forall f \ \exp(X_\sigma^{-1}, f, g) \in G_M^{inc} \quad (6)$$

From (6) it immediately follows that

$$O_M^{gr}(X_\sigma) \Rightarrow C_M^{gr}(X_\sigma^{-1}) \quad (7)$$

And from (7) it directly follows that $\Upsilon_M^{gr}$ can never evaluate a sentence as $b$ and so (as there clearly are sentences that are evaluated as $n$ by $\Upsilon_M^{gr}$) $\Upsilon_M^{gr}$ is a 3-valued theory of truth.

The principle that is underlying the 3-valuedness of $\Upsilon_M^{gr}$, i.e., (7), breaks down for $\Upsilon_M^{\bullet}$. For, we have

$$O_M^{\bullet}(X_\sigma) \neq C_M^{\bullet}(X_\sigma^{-1}) \quad (8)$$

The reason for this is that the set of expansions $U_M^{mi}$, which is open for $\bullet$, is not itself inverse. Hence, the fact that player I can force an expansion of $A_\sigma$ to end up in $U_M^{mi}$ implies that player II can force the expansion of $D_\sigma$ to end up in $(U_M^{mi})^{-1} = U_M^{mi}$. But the fact that player II can force the expansion of $D_\sigma$ to end up in $U_M^{mi}$ does not preclude the possibility that player I may as well be able to force $D_\sigma$ to end up in $U_M^{mi}$. Hence, $A_\sigma$ and $D_\sigma$ may both be open for $\bullet$. The previous remarks are illustrated by considering the following two expansions of the Truth teller:

$$A_{T(\tau)}, A_{T(\tau)}, A_{T(\tau)}, \ldots \quad D_{T(\tau)}, D_{T(\tau)}, D_{T(\tau)}, \ldots$$

The gr(oundedness) closure conditions thus induce $\mathcal{K}$, the 3$n$-valued version of the minimal fixed point. By invoking Lemma 3.9 and a further lemma (Lemma 3.10) we will show that the gr(oundedness) closure conditions induce $\mathcal{K}$, the 3$b$-valued version of the minimal fixed point.

### gr(oundedness) closure conditions:

$$\Upsilon_M^{inc} = G_M^{inc}$$
Lemma 3.10 If † satisfies SJC: $C_M^\dagger(\sigma) \iff \exists g \forall f \exp(X_\sigma, f, g) \in C_M^\dagger$

Proof The $\Leftarrow$ direction is trivial. To prove the $\Rightarrow$ direction, observe that, per definition

$$C_M^\dagger(\sigma) \Rightarrow \forall f \exists g \exp(X_\sigma, f, g) \in C_M^\dagger$$

(9)

Now let $Q$ be the set of all expansions of $X_\sigma$ that end up in $C_M^\dagger$ and let

$$Q' = \{ Y_\alpha \mid Y_\alpha \text{ occurs on some } \exp \in Q \text{ and is controlled by player II } \}$$

Now define, for each $Y_\alpha \in Q'$, the set $\text{Suc}(Y_\alpha)$ as

$$\text{Suc}(Y_\alpha) = \{ Z_\beta \mid Y_\alpha \text{ is succeeded by } Z_\beta \text{ on some } \exp \in Q \}$$

Let $h$ be a (choice) function that maps each $Y_\alpha \in Q'$ to an element of $\text{Suc}(Y_\alpha)$. Fix an arbitrary strategy $g'$ for player II. We define a strategy $g$ for player II as follows:

$$g(Y_\alpha) = \begin{cases} 
    h(Y_\alpha), & Y_\alpha \in Q' \\
    g'(Y_\alpha), & Y_\alpha \notin Q';
\end{cases}$$

From (9) and the fact that † satisfies SJC, it readily follows that the constructed strategy $g$ is such that $\forall f \exp(X_\sigma, f, g) \in C_M^\dagger$. \square

Proposition 3.11 $\forall \sigma = \overline{\mathcal{H}}$

Proof Let $M$ be a ground model. Observe that, as $\forall^\mathcal{L} = \mathcal{H}$, it suffices to show that for any $\sigma \in \text{Sen}(L_\mathcal{L})$:

\begin{itemize}
    \item[i] $\forall^M_M(\sigma) = a \iff \forall^M_M(\sigma) = a$
    \item[ii] $\forall^M_M(\sigma) = d \iff \forall^M_M(\sigma) = d$
\end{itemize}

i $\Rightarrow$ Suppose that $\forall^M_M(\sigma) = a$, i.e., that $O_M^\forall(A_\sigma)$ and $C_M^\forall(D_\sigma)$. As $O_M^\forall \subseteq O_M^\forall$, $O_M^\forall(A_\sigma)$ implies that $O_M^\forall(A_\sigma)$. Further, $O_M^\forall(A_\sigma)$ means that player I has a strategy that ensures that the expansion of $A_\sigma$ will end up in $G_M^\forall$. As the inverse of $G_M^\forall$ is $G_M^\forall$, this implies, via Lemma 3.9, that player I has a strategy $g$ that ensures that the expansion of $D_\sigma$ will end up in $G_M^\forall$. But this means that player I does not have a strategy that ensures that the expansion of $D_\sigma$ will end up in $\text{EXP}_M - G_M^\forall = O_M^\forall$. Thus, $C_M^\forall(D_\sigma)$. Together with the already established $O_M^\forall(A_\sigma)$, we thus have $\forall^M_M(\sigma) = a$.

i $\Leftarrow$ Suppose that $\forall^M_M(\sigma) = a$, i.e., that $O_M^\forall(A_\sigma)$ and that $C_M^\forall(D_\sigma)$. As $O_M^\forall \subseteq O_M^\forall$, $C_M^\forall(D_\sigma)$ implies that $C_M^\forall(D_\sigma)$. Further, from $C_M^\forall(D_\sigma)$ and as the $\forall(\text{boundedness})$ closure condition satisfies SJC, it follows from Lemma 3.10 that player II has a strategy $g$ that ensures that the expansion of $D_\sigma$ will end up in $G_M^\forall$. This implies, via Lemma 3.9 and as $G_M^\forall$ is the inverse of $G_M^\forall$, that player I has a strategy $f$ that ensures that the expansion of $A_\sigma$ will end up in $G_M^\forall$. Hence $O_M^\forall(A_\sigma)$. Together with the already established $C_M^\forall(D_\sigma)$ we thus have $\forall^M_M(\sigma) = a$.

ii Just like i.

In order to get a better grip on the role of SJC in the construction of SK theories, it is instructive to compare the $\star$ closure conditions with the $\circ$ closure conditions, that are defined below. Before we define the the $\circ$ closure conditions, observe that the $\star$ closure conditions allow for the following, equivalent, definition:

\textit{\textbf{$\star$ closure conditions}}: $C_M^\star = G_M^{\text{inc}} \cup U_M^{\text{vic}}$
This reformulation is convenient as it clearly lays bare the distinction with the ◇ closure conditions:

\[
\text{◇ closure conditions: } C_M^\diamond = C_M^{\text{op}} \cup \{ \exp \mid \exists \sigma \in \text{Sen}(L_T): A_\sigma, D_\sigma \text{ on } \exp \}
\]

So the only difference between the ◊ closure conditions and the ◇ closure conditions is that the former classifies all expansions that contain \(A_\sigma\) and \(D_\sigma\) in a cycle as closed, whereas the latter does away with the condition of cyclicity: whenever an expansion contains an “AD clash” it is closed, whether or not this clash occurs in a cycle. To illustrate the difference between the ◊ and the ◇ closure conditions, we consider the following expansion of a denial of the Tautologyteller (cf. Definition 2.1):

\[
D_{T(\eta)\lor\neg T(\eta)}; D_{\neg T(\eta)}; A_{T(\eta)}; A_{T(\eta)\lor\neg T(\eta)}; A_{T(\eta)}; A_{T(\eta)\lor\neg T(\eta)}; \ldots
\]

(10)

This expansion is open according to the ◊ closure conditions—as the “AD clash” does not occur in a cycle—while it is closed according to the ◇ closure conditions. The successor expansion of (10), however, is open according to the ◇ closure conditions, which establishes that these closure conditions do not satisfy SJC.

\(\forall\diamond\) defines a 4-valued theory of truth of which it can be shown\(^{12}\) that it is not an SK\(_3\) theory. This raises the question whether satisfying SJC is, besides a sufficient condition, also a necessary condition for closure conditions to induce an SK valuation that respects the identity of truth. The answer to that question is ‘no’, as testified by the following proposition.

**Proposition 3.12 Violating SJC and inducing an SK\(_3\) theory.**

**Proof** The * closure conditions, stated below, violate SJC while they define a 3-valued SK theory of truth. In the definition of the * closure conditions, \(c\) is an arbitrary non-quotation constant of \(L_T\).

\[O_M^* = G_M^{\text{op}} \cup \{ \exp \mid A_{T(c)}\lor\neg T(c) \text{ or } D_{T(c)} \text{ on } \exp \text{ and } I(c) = T(c) \}
\]

(11)

The * closure conditions are a (minimal) modification of the (gr)oundedness closure conditions. According to the * closure conditions, the expansions in \(G_M^{\text{op}}\) are open and, besides those, all (and only) the expansions that contain \(A_{T(c)}\lor\neg T(c)\) or \(D_{T(c)}\) for some \(c\) such that \(I(c) = T(c)\) are open. A little reflection shows that this ensures that \(\forall_M^*\) is just like \(\forall_M^{gr}\) apart from a valuation of Truthtellers—i.e., sentences of form \(T(c)\) such that \(I(c) = T(c)\)—and compounds of Truthtellers. In particular, with \(T(\tau)\) a Truthteller, we have

\[\forall_M^*(T(\tau)) = a, \quad \forall_M^*(T(\tau) \lor \neg T(\tau)) = a\]

Being a minimal modification of \(\forall_M^{gr}\), \(\forall_M^*\) is an SK\(_3\) theory. However, the * closure conditions violate SJC, which is easily seen by inspecting the following expansion:

\[A_{T(\tau)\lor\neg T(\tau)}; A_{T(\tau)}; A_{T(\tau)}; A_{T(\tau)}; \ldots\]

Indeed, this expansion is open according to * closure conditions as it contains \(A_{T(\tau)\lor\neg T(\tau)}\) and as \(I(\tau) = T(\tau)\). Its successor expansion, which does not contain \(A_{T(\tau)\lor\neg T(\tau)}\) or \(D_{T(\tau)}\), is closed and so the * closure conditions violate SJC while they induce an SK\(_3\) theory.

Thus, Proposition 3.12 testifies that the first stable judgement theorem cannot be read in the converse direction. However, our *second stable judgement theorem* comes close to a converse reading of the first stable judgement theorem.
3.5 The second stable judgement theorem In this section, we present the second stable judgement theorem, which states that any SK valuation that respects the identity of truth can be induced from a closure condition that satisfies SJC. A corollary of the second stable judgement theorem states that any SK fixed point can be induced from a closure condition that satisfies SJC and WRC.

Before we state the second stable judgement theorem and its corollary, we define the notion of the correctness of an AD sentence with respect to a (2-, 3- or 4-valued) valuation $V_M$.

**Definition 3.13 $V_M$ correctness**

Let $V_M$ be a (2-, 3- or 4-valued) valuation for $L_T$ whose range $V$ is such that \{a, d\} $\subseteq V \subseteq \{a, b, n, d\}$. The notion of $V_M$ correctness, applicable to AD sentences, is defined as follows.

\[ X_\sigma \text{ is } V_M \text{ correct } \iff (X = A, V_M(\sigma) \in \{a, b\}) \text{ or } (X = D, V_M(\sigma) \in \{d, b\}) \]

Intuitively, an AD sentence $X_\sigma$ is $V_M$ correct iff its judgement (Assertible or Deniable) with respect to $\sigma$ is correct from the standpoint of $V_M$.

**Theorem 3.14 Second stable judgement theorem**

Let $M$ be a ground model and let $V_M$ be a 2-, 3- or 4-valued Strong Kleene valuation of $L_T$ that respects the identity of truth. Then there is a closure condition $\dagger$ that satisfies SJC and such that $V_M^{\dagger} = V_M$.

**Proof** Let $V_M$ be a 2-, 3- or 4-valued SK valuation of $L_T$ that respects the identity of truth. Using the notion of $V_M$ correctness, we define a closure condition $\dagger = \{O_M^{\dagger}, C_M^{\dagger}\}$ and will show that $\dagger$ satisfies SJC and is such that $V_M^{\dagger} = V_M$. Let $\exp = \{y_n\}_{n \in \mathbb{N}}$ be an arbitrary expansion in $\exp_M$. We let

\[ \exp \in O_M^{\dagger} \iff \exists n, \forall m > n : y_m \text{ is } V_M \text{ correct} \]  

(12)

It is clear, from the “limit behavior definition” of $\dagger$, that $\dagger$, satisfies SJC. Note that, in order to show that $V_M^{\dagger} = V_M$, it suffices to show that

\[ X_\sigma \text{ is } V_M \text{ correct } \iff \exists \bar{f} \in \mathcal{F} \forall g \in \mathcal{G} : \exp(X_\sigma, \bar{f}, g) \in O_M^{\dagger} \]

$\Rightarrow$ Suppose that $X_\sigma$ is $V_M$ correct. Observe that, from the fact that $V_M$ is SK and respects the identity of truth, we have

\[ X_\sigma \text{ is controlled by player I } \Rightarrow \exists Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ correct} \]

\[ X_\sigma \text{ is controlled by player II } \Rightarrow \forall Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ correct} \]

From these two observations, it readily follows that if we start from an $X_\sigma$ that is $V_M$ correct, player I has a strategy, say $f$, that ensures that, for every $g \in \mathcal{G}$, all the terms of $\exp(X_\sigma, f, g)$ are $V_M$ correct. Hence, player I can ensure that the expansion of $X_\sigma$ ends up in $O_M^{\dagger}$.

$\Leftarrow$ Suppose that $X_\sigma$ is $V_M$ incorrect. Observe that, from the fact that $V_M$ is SK and respects the identity of truth, we have

\[ X_\sigma \text{ is controlled by player I } \Rightarrow \forall Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ incorrect} \]

\[ X_\sigma \text{ is controlled by player II } \Rightarrow \exists Y_\alpha \in \Pi(X_\sigma) : Y_\alpha \text{ is } V_M \text{ incorrect} \]

From these two observations, it readily follows that if we start from an $X_\sigma$ that is $V_M$ incorrect, player II has a strategy, say $g$, that ensures that, for every $f \in \mathcal{F}$, all the terms of $\exp(X_\sigma, f, g)$ are $V_M$ incorrect. Hence, player II can ensure that the
expansion of $X_\sigma$ ends up in $C^I_M$, from which it follows that player $I$ cannot ensure that the expansion of $X_\sigma$ ends up in $O^I_M$.

**Corollary 3.15  Second stable judgement corollary**

Let $M$ be a ground model and let $V_M$ be an $SK$ fixed point over $M$. Then there is a closure condition $\dagger$ that satisfies SJC and WRC and such that $\mathcal{V}^\dagger_M = V_M$.

**Proof**  As an $SK$ fixed point is an $SK$ valuation that respects the identity of truth and the world, the closure condition $\dagger$ that is defined in the proof of the second stable judgement theorem satisfies SJC and WRC and is such that $\mathcal{V}^\dagger_M = V_M$.

So, in order to induce, say, Kripke’s $SK$ maximal intrinsic fixed point $\mathcal{X}^+$, via the method of closure games, we may define closure conditions, via (12), in terms of $\mathcal{X}^+_M$ correctness. Closure conditions for $\mathcal{X}^+$ that are defined as such are parasitic on Kripke’s framework for truth in a way that the gr(oundedness), $\overline{gr}(roundedness)$, $\spadesuit$ and $\heartsuit$ closure conditions are not. We say that closure conditions for $\mathcal{X}^+$ that are defined via (12) are cheating closure conditions, whereas the gr(oundedness), $\overline{gr}(roundedness)$, $\spadesuit$ and $\heartsuit$ are species of non-cheating closure conditions. As $\mathcal{X}^+$ is an interesting theory of truth, it would be bad news for the method of closure games, as a framework for truth, if it had to rely, for the definition of $\mathcal{X}^+$, on notions that are borrowed from an alternative framework. Luckily, the method of closure games does have access to $\mathcal{X}^+$ via notions that are not borrowed from an alternative framework. In the next section, we see how this works out.

### 4 Assertoric semantics

**4.1 Defining assertoric semantics**  In this section we present assertoric semantics, which is a slight modification of the method of closure games. Whereas the method of closure games induces $L_T$ valuations by putting closure conditions on sequences of signed sentences (expansions), assertoric semantics does so by putting closure conditions on branches, which are sets of signed sentences that are defined in terms of expansions.

For any expansion $\exp$, $\exp$ will denote the set of terms of $\exp$. For any AD sentence $X_\sigma$ and strategy $f$ of player $I$, $B_f(X_\sigma)$ denotes the set of terms that occur on some expansion of $X_\sigma$ relative to $f$. We will call $B_f(X_\sigma)$ the branch of $X_\sigma$ that is induced by $f$. To be sure, $B_f(X_\sigma)$ is defined as follows:

$$B_f(X_\sigma) = \bigcup_{g \in \mathcal{F}} \exp(X_\sigma, f, g)$$

We will use $\text{Branch}_M$ to denote the set of all branches relative to ground model $M$. The (assertoric) tree of $X_\sigma$, $\Sigma^\mathcal{F}_X$, is the set of all its branches:

$$\Sigma^\mathcal{F}_X = \{B_f(X_\sigma) \mid f \in \mathcal{F}\}$$

Branches are judged to be open or closed relative to closure conditions that are applicable to branches; a branch closure condition $\infty = \{O^I_M, C^I_M\}$ is a bipartition of $\text{Branch}_M$. An assertoric tree $\Sigma^\mathcal{F}_X$ is said to be open$^1$ in $M$ just in case it contains a branch that is open$^1$ in $M$, i.e., just in case $B_f(X_\sigma) \in O^I_M$ for some $B_f(X_\sigma) \in \Sigma^\mathcal{F}_X$. We write $O^I_M(X_\sigma)$ just in case $\Sigma^\mathcal{F}_X$ is open$^1$ in $M$, and $C^I_M(X_\sigma)$ if not $O^I_M(X_\sigma)$. In this sense, branch closure conditions induce closure conditions for AD sentences. These
closure conditions can be used to define an $L_T$ valuation $\mathcal{V}_M^V$ in the expected manner. That is:

$$\mathcal{V}_M^V(\sigma) = \begin{cases} 
\text{a} & O_M^\sharp(A_\sigma) \text{ and } C_M^\sharp(D_\sigma); \\
\text{b} & O_M^\sharp(A_\sigma) \text{ and } O_M^\sharp(D_\sigma); \\
\text{n} & C_M^\sharp(A_\sigma) \text{ and } C_M^\sharp(D_\sigma); \\
\text{d} & C_M^\sharp(A_\sigma) \text{ and } O_M^\sharp(D_\sigma). 
\end{cases}$$

(13)

4.2 Inducing familiar theories of truth via assertoric semantics In this paper, we will only be concerned with two closure conditions for branches: the tolerant and strict closure condition. A branch $B$ is tolerantly closed in $M$, i.e., contained in $C_M^{\text{tol}}$, just in case

$B$ contains $X_\sigma$ with $X_\sigma \in A_\sigma M$ and $X_\sigma \not\in \text{WM}$

A branch $B$ is strictly closed in $M$, i.e., contained in $C_M^{\text{strict}}$, just in case $B$ is tolerantly closed in $M$ or

$B$ contains both $A_\sigma$ and $D_\sigma$

The tolerant closure condition induces a familiar theory of truth.

Proposition 4.1 $\mathcal{V}^{\text{tol}} = \mathcal{K}$

Proof Let $M$ be a ground model. Clearly, it suffices to show that, for each $AD$ sentence $X_\sigma$

$$\exists B \in \Sigma_X^n : B \in O_M^{\text{tol}} \iff \exists f \in \mathcal{F} \forall g \in \mathcal{G} : \exp(X_\sigma, f, g) \in O_M^{\text{tol}}$$

(14)

Now (14) follows immediately from a comparison of the tolerant closure condition for branches with the $\mathcal{G}$-boundedness) closure condition for expansions and from the fact that $\mathcal{V}^{\text{tol}} = \mathcal{K}$. $\square$

$\mathcal{V}^{\text{strict}}$, the theory of truth that is induced by the strict closure condition for branches, is also a—albeit less—familiar theory of truth. As we will show next, $\mathcal{V}^{\text{strict}}$ is equivalent to Kripke’s “modal fixed point valuation” $\mathcal{K}_M^\sharp$, which he defined implicitly by quantifying over $F^{\text{pol}}(M)$: for instance, Kripke classified the Liar as a paradoxical sentence as $(N)$ in which it evaluates as $t$ and there is no fixed point in which it evaluates as $d$. More generally, Kripke used these quantifications (cf. pages 708 and 709 of [4]) over $F^{\text{pol}}(M)$ to draw distinctions between the Liar, the Truth, the Tautology, and the Contradiction (cf. Definition 2.1). These distinctions are captured by $\mathcal{K}_M^\sharp$, which is defined by quantifying over $F^{\text{pol}}(M)$ as follows:

- $\mathcal{K}_M^\sharp(\sigma) = \text{a} \iff$ for some $V_M : V_M(\sigma) = \text{a}$ and for no $V_M : V_M(\sigma) = \text{d}$
- $\mathcal{K}_M^\sharp(\sigma) = \text{b} \iff$ for some $V_M : V_M(\sigma) = \text{a}$ and for some $V_M : V_M(\sigma) = \text{d}$
- $\mathcal{K}_M^\sharp(\sigma) = \text{n} \iff$ for no $V_M : V_M(\sigma) = \text{a}$ and for no $V_M : V_M(\sigma) = \text{d}$
- $\mathcal{K}_M^\sharp(\sigma) = \text{d} \iff$ for no $V_M : V_M(\sigma) = \text{a}$ and for some $V_M : V_M(\sigma) = \text{d}$

Although $\mathcal{K}_M^\sharp$ respects the ground model $M$ and the identity of truth, $\mathcal{K}_M^\sharp$ is not an $S_K^4$ valuation, which is testifiable by the following observations:

$\mathcal{K}_M^\sharp(T(\tau)) = \mathcal{K}_M^\sharp(\neg T(\tau)) = \text{b}, \quad \mathcal{K}_M^\sharp(T(\tau) \lor \neg T(\tau)) = \text{a}$

In order to prove that $\mathcal{V}^{\text{strict}} = \mathcal{K}^\sharp$, we need some definitions, which are all modifications of notions defined, amongst others, in Fitting [1].
**Definition 4.2  Saturated sets, upwards closure**

Let $S$ be a set of AD sentences. We say that $S$ is *downwards saturated* just in case:

player I controls $X_\sigma \Rightarrow (X_\sigma \in S \Rightarrow \Pi(X_\sigma) \cap S \neq \emptyset)$

player II controls $X_\sigma \Rightarrow (X_\sigma \in S \Rightarrow \Pi(X_\sigma) \subseteq S)$

This notion of an *upwards saturated* set is defined dually. That is, $S$ is upwards saturated just in case:

player I controls $X_\sigma \Rightarrow (X_\sigma \in S \Leftarrow \Pi(X_\sigma) \cap S \neq \emptyset)$

player II controls $X_\sigma \Rightarrow (X_\sigma \in S \Leftarrow \Pi(X_\sigma) \subseteq S)$

Every set of AD sentences $S$ has an *upwards closure* $S^\uparrow$, i.e., a smallest set of AD sentences which extends $S$ and which is upwards saturated$^{17}$.

**Definition 4.3  FP$^n(M)$ sets and associated valuations**

Let $S$ be a set of AD sentences. We say that $S$ is an FP$^n(M)$ set just in case:

1. $\forall \sigma \in \mathcal{S}en(L_T): A_\sigma \in S \Rightarrow D_\sigma \not\in S$.
2. $S$ is downwards and upwards saturated.
3. $w_M \subseteq S$.

An FP$^n(M)$ set $S$ is a notational variant of the associated fixed point valuation, $V^S_M: \mathcal{S}en(L_T) \rightarrow \{a, n, d\}$:

i) $A_\sigma \in S \Leftrightarrow V^S_M(\sigma) = a$.
ii) $D_\sigma \in S \Leftrightarrow V^S_M(\sigma) = d$.
iii) $\{A_\sigma, D_\sigma\} \cap S = \emptyset \Leftrightarrow V^S_M(\sigma) = n$.

On the other hand, every fixed point valuation $V_M \in \text{FP}^n(M)$ corresponds via i), ii), and iii) to an FP$^n(M)$ set $S$.

Before we prove that $\forall^{\text{strict}} = \mathcal{X}^4$ it is instructive to comment on our proof strategy, which is a modification of the soundness and completeness proofs for signed tableaux systems. Consider the assertoric rules for $\land$ and $\neg$ for a propositional language $\mathcal{L}_P$ under the usual closure conditions: a branch$^{18}$ is closed just in case it contains, for some sentence $\sigma$ of $\mathcal{L}_P$, both $A_\sigma$ and $D_\sigma$ and a tableau for $X_\sigma$ is closed just in case all its branches are closed. This specifies a sound and complete signed tableau proof system with respect to the classical semantics of $\mathcal{L}_P$: a sentence $\sigma$ of $\mathcal{L}_P$ is true in every $\mathcal{L}_P$ valuation just in case $D_\sigma$ has a closed tableau. Soundness is proved by observing that if $D_\sigma$ has a closed tableau, there is no $\mathcal{L}_P$ valuation in which $\sigma$ is false. Completeness is proved by showing that if $D_\sigma$ does not have a closed tableau, we can take an open branch and transform it into an $\mathcal{L}_P$ valuation which renders $\sigma$ false.

We use our branches and assertoric trees to induce semantic valuations; our strict closure conditions are defined relative to a ground model $M$. The role that is played by the classical valuation in the $\mathcal{L}_P$ case is, in our case, played by a fixed point of FP$^n(M)$. If all the branches of $\mathcal{T}_A^\sigma$ are strictly closed, there is no fixed point in which $\sigma$ is evaluated as $a$. Similarly, if all the branches of $\mathcal{T}_B^\sigma$ are strictly closed, there is no fixed point in which $\sigma$ is evaluated as $d$. On the other hand, if $\mathcal{T}_A^\sigma$ has a strictly open branch, we can convert this branch into a fixed point which evaluates $\sigma$ as $a$. Similarly for the case when $\mathcal{T}_B^\sigma$ has a strictly open branch. Let us now turn to the proof that makes these remarks precise.
From Closure Games to Strong Kleene Truth

4.4 $\forall^{\text{strict}} = \mathcal{K}^4$

Proof Let $M$ be a ground model. Let $B$ be a branch of $A_\sigma$ that is strictly open in $M$. Then, $(B \cup w_M)^+$, i.e., the upwards closure of $B \cup w_M$, is an $FP^M(M)$ set that contains $A_\sigma$. From this, it follows that

$$O^\text{strict}_M(A^\sigma) \Rightarrow \exists V_M \in FP^M(M) : V_M(\sigma) = a$$

And, similarly, we get that

$$O^\text{strict}_M(D^\sigma) \Rightarrow \exists V_M \in FP^M(M) : V_M(\sigma) = d$$

On the other hand, let $\sigma \in \text{Sen}(L_T)$ and let $V_M \in FP^M(M)$ be such that $V_M(\sigma) = a$. Let $S$ be the $FP^M(M)$ set associated with $V_M$. Per definition, $A_\sigma \subseteq S$. Let $f$ be any strategy for player $I$ such that, for every $X_\sigma \subseteq S$ that is controlled by player $I$, $f(X_\sigma) \subseteq S$. It follows that $B_I(A_\sigma)$ is strictly open in $M$. Similar remarks apply to $V_M(\sigma) = d$ and $D_\sigma$. Hence, we get that

$$\exists V_M \in FP^M(M) : V_M(\sigma) = a \Rightarrow O^\text{strict}_M(A^\sigma)$$

$$\exists V_M \in FP^M(M) : V_M(\sigma) = d \Rightarrow O^\text{strict}_M(D^\sigma)$$

From the four established equations, it follows that $\forall^{\text{strict}} = \mathcal{K}^4_M$. $\square$

Although $\forall^{\text{strict}}$ is not defined via the method of closure games, it is clearly defined using only notions that “belong” to the method of closure games. In the next subsection, we will use $\forall^{\text{strict}}$ to define closure conditions that define $\mathcal{K}^+$ and $\overline{\mathcal{K}}^+$. Doing so, we obtain a definition of $\mathcal{K}^+$ that is, in an important sense, not parasitic on Kripke’s framework for truth.

4.3 Using $\forall^{\text{strict}}$ to define $\mathcal{K}^+$ and $\overline{\mathcal{K}}^+$ We will prove that $\mathcal{K}^+$ can be induced from closure conditions that are defined in terms of $\forall^{\text{strict}}$. To do so, we define the notion of strong correctness in terms of $\forall^{\text{strict}}_M$. We say that $X_\sigma$ is strongly correct (in a ground model $M$) just in case

$$(X = A \text{ and } \forall^{\text{strict}}_M(\sigma) = a) \text{ or } (X = D \text{ and } \forall^{\text{strict}}_M(\sigma) = d)$$

We use the strong correctness to define the $+$ closure condition (for expansions) which, as we will eventually prove, induces $\mathcal{K}^+$. With $\text{exp} = \{y_n\}_{n \in \mathbb{N}}$, we let

$$\text{exp} \in O^+_M \iff \exists \forall n : y_n \text{ is strongly correct}$$

We will show that the valuation function induced by the strong closure condition, i.e., $\mathcal{F}_M^+$, is equal to $\mathcal{K}^+_M$. Before we do so, however, we first sketch the rationale of the definition of $\mathcal{K}^+_M$ in terms of strong $\forall^{\text{strict}}_M$ correctness.

For sure, if we have $\mathcal{K}^+_M(\sigma) = a$, we have $\forall^{\text{strict}}_M(\sigma) = a$. For, if $\mathcal{K}^+_M(\sigma) = a$, there is a (3-valued SK) fixed point that evaluates $\sigma$ as $a$ and also, there is no fixed point that evaluates $\sigma$ as $d$. Similarly, $\mathcal{K}^+_M(\sigma) = d$ implies that $\forall^{\text{strict}}_M(\sigma) = d$. The converses of these implications do not hold, however. For instance, we have

$$\forall^{\text{strict}}_M(\neg T(\lambda) \lor T(\tau)) = a \quad \mathcal{K}^+_M(\neg T(\lambda) \lor T(\tau)) = n$$

$$\forall^{\text{strict}}_M(\neg T(\tau) \land T(\tau)) = d \quad \mathcal{K}^+_M(\neg T(\tau) \land T(\tau)) = n$$

Although $A \rightarrow T(\lambda) \lor T(\tau)$ and $D \rightarrow T(\tau) \lor T(\tau)$ are strongly correct, none of their immediate $AD$ subsentences is strongly correct. This ensures, as is readily noticed, that $\mathcal{K}^+_M(\neg T(\lambda) \lor T(\tau)) = \mathcal{K}^+_M(\neg T(\tau) \land T(\tau)) = n$, mimicking the judgement of $\mathcal{K}^+_M$...
with respect to these sentences. More generally, the definition of $\gamma^+_M$ ensures that, for AD sentences that are “unstable” strongly correct—i.e., ultimately, they depend on a combination of AD sentences that are not strongly correct—player I does not have a strategy that ensures that his expansion ends up in $O^+_M$. In order to prove that $\gamma^+_M = \mathcal{A}^+_M$, we will evoke the following three lemmas.

**Lemma 4.5** **strict-openess is preserved downwards in assertoric trees**

By the phrase ‘strict-openess is preserved downwards in assertoric trees’, we mean that

- player I controls $X_\alpha \Rightarrow (O^\strict_M(X_\alpha) \Rightarrow \exists Y_\alpha \in \Pi(X_\alpha) : O^\strict_M(Y_\alpha))$
- player II controls $X_\alpha \Rightarrow (O^\strict_M(X_\alpha) \Rightarrow \forall Y_\alpha \in \Pi(X_\alpha) : O^\strict_M(Y_\alpha))$

**Proof** This follows immediate from an inspection of the strict closure conditions and the observation that the branches that constitute the tree of an immediate AD subsentence of $X_\alpha$ are subsets of the branches that constitute the tree of $X_\alpha$.

**Lemma 4.6** $\gamma^+_M : \text{Sen}(L_\mathcal{T}) \rightarrow \{a, n, d\}$ is an SK$_3$ theory

**Proof** It is clear that the strong closure conditions satisfy SJC and WRC and so, by the (corollary to the) first stable judgement Theorem, they define a Strong Kleene theory. The point of this lemma then, is to show that $\gamma^+_M$ is 3-valued. To do so, we proceed as in the proof of Lemma 3.9. Suppose that $O^\strict_M(A_\sigma)$ and let $f$ be a strategy of player I that ensures that the expansion of $A_\sigma$ ends up in $O^+_M$. The mirror strategy of $f$, $g_f$ (see Lemma 3.9), testifies that $C^+_M(D_\sigma)$.

Our proof of the fact that $\gamma^+_M = \mathcal{A}^+_M$ will exploit a further lemma, which invokes the notion of a **totally strongly correct expansion**. An expansion is said to be totally strongly correct just in case all its terms are strongly correct. Here is the lemma:

**Lemma 4.7** $\forall g \in \mathcal{G} : \exp(X_\sigma, f', g) \in O^+_M \Leftrightarrow \forall g \in \mathcal{G} : \exp(X_\sigma, f', g)$ is **totally strongly correct**

**Proof** The right to left direction is trivial. For the converse direction, let $f'$ be a strategy testifying that $O^\strict_M(X_\alpha)$, i.e.,:

$$\forall g \in \mathcal{G} : \exp(X_\sigma, f', g) \in O^+_M$$

Let $g' \in \mathcal{G}$. We have to show that $\exp' = \exp(X_\sigma, f', g')$ is totally strongly correct. As $\exp' \in O^+_M$, $\exp'$ contains a first strongly correct term (after which all other terms are strongly correct). We will prove by contraposition that this first term is equal to $X_\sigma$. Thus, assume that $\exp'$ contains a first strongly correct term and that this term has a predecessor on $\exp'$ that is not strongly correct. Without loss of generality, that the first strongly correct term has form $A_\alpha$, the case where its form is $D_\alpha$ being similar. The predecessor of $A_\alpha$ on $\exp$ has one of the following six forms:

$$D_\alpha, A_{\alpha/\beta}, A_{\alpha/\beta_1}, A_{\max(\prec)}, A_{\exists \phi(x)}, A_{T(\pi)}$$

We only prove the claim for the cases where the predecessor of $A_\alpha$ is $A_{\alpha/\beta}$ or $A_{\alpha/\beta_1}$, as the other four cases are either trivial or similar to the two cases that we will discuss.

**Predecessor of $A_\alpha$ is $A_{\alpha/\beta}$**. As $A_\alpha$ is strongly correct, we have $\gamma^\strict_M(\alpha) = a$. Hence, there is a (3-valued SK) fixed point in which $\alpha$ is evaluated as $a$ and no fixed point in which $\alpha$ is evaluated as $d$. In the fixed point in which $\alpha$ is evaluated
as $a$, $\alpha \lor \beta$ is also evaluated as $a$. Thus, $\nu^\text{strict}_M(\alpha \lor \beta) = a$. This gives a contradiction with the assumption that $A_\alpha$ is the first strong $\nu^\text{strict}_M$ correct element on $\exp'$. Thus, suppose that $\nu^\text{strict}_M(\alpha \lor \beta) = b$. Per definition of $\nu^\text{strict}_M$, we get $O^\text{strict}_M(D_{\alpha \lor \beta})$. From Lemma 4.5, we get that $O^\text{strict}_M(D_\alpha)$ and $O^\text{strict}_M(D_\beta)$. From $O^\text{strict}_M(D_\alpha)$ it follows, by Theorem 4.4, that there is a fixed point in which $\alpha$ is evaluated as $d$. This gives a contradiction with the strong correctness of $A_\alpha$.

**Predecessor of $A_\alpha$ is $A_{\alpha \lor \beta}$.** As $A_\alpha$ is strongly correct, we have $\nu^\text{strict}_M(\alpha) = a$. Further, strategy $f'$ (by considering the mirror strategy of $f'$ as in the proof of Lemma 4.6) testifies that $\nu^\text{strict}_M(\alpha \land b) = \nu^\text{strict}_M(\alpha) = \nu^\text{strict}_M(b) = a$. From the fact that $\nu^\text{strict}_M(\alpha \land b) = a$, it follows that there is a 3-valued fixed point (namely, $\nu^\text{strict}_M$) in which $\alpha \land b$ is valued as $a$. Hence, from Theorem 4.4, it follows that $\nu^\text{strict}_M(\alpha \land b) \in \{a, b\}$. Suppose that $\nu^\text{strict}_M(\alpha \land b) = a$. This gives a contradiction with the assumption that $A_\alpha$ is the first strongly correct element on $\exp'$. Thus, suppose that $\nu^\text{strict}_M(\alpha \land b) = b$. From Lemma 4.5, we get that $O^\text{strict}_M(D_\beta)$. Further, from $\nu^\text{strict}_M(\alpha) = a$ it follows, per definition, that $C^\text{strict}_M(D_\alpha)$. Similarly, from $\nu^\text{strict}_M(\alpha \land b) = b$ we get, per definition, that $O^\text{strict}_M(D_{\alpha \land \beta})$. From $O^\text{strict}_M(D_{\alpha \land \beta})$ and $C^\text{strict}_M(D_\alpha)$ it follows that $O^\text{strict}_M(D_\beta)$ and so $\nu^\text{strict}_M(b) = b$. Hence $A_\beta$ is not strongly correct. Now, let $g'' \in \mathcal{G}$ be the strategy that is defined just like $g'$ except for the fact that $g'(A_{\alpha \land \beta}) = A_\alpha$, whereas $g''(A_{\alpha \land \beta}) = A_\beta$. Let $\exp'' = \exp(X_\sigma, f', g'')$ be the expansion of $X_\sigma$ induced by $f'$ and $g''$ and note that $A_{\alpha \land \beta}$ occurs on $\exp''$. Let $Y_f$ be the first element controlled by player $I$ that occurs on $\exp''$ after $A_{\alpha \land \beta}$ such that $|\Pi(Y_f)| > 1$. If there is no such element if follows, from Lemma 4.5, that for every element $Z_\theta$ on $\exp''$, we have $\nu^\text{strict}_M(\exp''(Z_\theta)) = b$. Observe that this contradicts with the assumption that strategy $f'$ guarantees that for every $g$, $\exp(X_\sigma, f', g)$ ends up in $O^+_M$. Thus, let $Y_f$ be as indicated. From Lemma 4.5, it follows that $\Pi(Y_f)$ contains at least one element, say $Y_\delta$, such that $O^\text{strict}_M(Y_\delta)$. Moreover, from the definition of $f'$, it follows that $f'$ has to pick a $Y_\delta \in \Pi(Y_f)$ such that $O^\text{strict}_M(Y_\delta)$. For suppose not, i.e., suppose that $f'(Y_f) = Y_{\delta'}$ such that $C^\text{strict}_M(Y_{\delta'})$. According to Theorem 4.4, this means that there is no 3-valued fixed point that contains $Y_{\delta'}$. On the other hand, from the definition of $f'$ and the assumption that $f'(Y_f) = Y_{\delta'}$, it follows that there is a 3-valued fixed point (namely, $\nu^\text{strict}_M$) that contains $Y_{\delta'}$. Thus, $f'(Y_f) = Y_\delta$ for some $Y_\delta$ such that $O^\text{strict}_M(Y_\delta)$. From Lemma 4.5, the fact that $\nu^\text{strict}_M(\alpha \land b) = b$ and the fact that $Y_f$ is the first element on $\exp''$ after $A_{\alpha \land \beta}$ for which player $I$ has to make a genuine choice, it follows that $O^\text{strict}_M(Y_\delta^{-1})$. Hence, we have $\nu^\text{strict}_M(\delta) = b$. And so $Y_\delta$ is not strongly correct. We are now back where we started, with $\delta$ playing the role of $\beta$. We can repeat the argument, by looking at the first element that occurs on $\exp''$ after $Y_f$ for which player $I$ has to make a genuine choice. By a similar argument, $f'$ cannot allot a strongly correct element to it. Hence, $f'$ does not guarantee that for every $g$, $\exp(X_\sigma, f', g)$ ends up in $O^+_M$.

Before we (finally) show that $\nu^+ = \mathcal{K}^+$, we first recall the definition of $\mathcal{K}^+_M$ in terms of the $\mathcal{K}^+$ closure conditions that are associated with the second stable judgement theorem. With $\exp = \{y_n\}_{n \in \mathbb{N}}$, these closure conditions are defined as follows:

$$\exp \in O^\mathcal{K}_M \iff \exists n \forall m > n : y_m \text{ is } \mathcal{K}^+_M \text{ correct}$$

**Theorem 4.8**

$$\nu^+ = \mathcal{K}^+$$
Proof Let $M$ be a ground model. It suffices to show that, for every $AD$ sentence $X_\sigma$, we have

$$O_M^{\pm}(X_\sigma) \Leftrightarrow O_M^{\pm}(X_\sigma)$$

The left to right direction is immediate from the definition of $O_M^{\pm}$ and $O_M^{\pm}$. Thus, assume that $O_M^{\pm}(X_\sigma)$. This means that there exists an $f \in F$ such that for every $g \in \mathcal{G}$, $\exp(X_\sigma, f, g) \in O_M^{\pm}$. By Lemma 4.7, this means that every term that occurs on an expansion of $X_\sigma$ that is induced by $f$, is strongly correct. Hence, all elements of $B_f(X_\sigma)$, the branch of $X_\sigma$ as induced by $f$, are strongly correct. From this, it follows that the (3-valued Strong Kleene) fixed point valuation induced by $B_f(X_\sigma)$, i.e., by the upwards closure of $B_f(X_\sigma)$, is compatible (see Section 2) with every fixed point valuation over $M$ and hence is an intrinsic fixed point valuation, i.e., a member of $\mathbf{P}(M)$ (see Definition 2.6). With $S$ the $\mathbf{FP}^n(M)$ set corresponding to $\mathcal{K}_M$, we get that $B_f(X_\sigma)^{\mathcal{K}} \subseteq S$, as $\mathcal{K}_M$ is maximal intrinsic. From $B_f(X_\sigma) \subseteq S$, it follows that $O_M^{\pm}(X_\sigma)$. \hfill \square

We end this section by defining a closure condition that induces $\mathcal{K}^-$, the 3b-version of the maximal intrinsic fixed point. To do so, we first define the notion of an $AD$ sentence being strongly incorrect. We say that $X_\sigma$ is strongly incorrect (in a ground model $M$) just in case

$$(X = A and \forall \sigma^{\text{strict}}(\sigma) = d) \text{ or } (X = D and \forall \sigma^{\text{strict}}(\sigma) = a)$$

The $\mathcal{K}$ closure conditions are defined in terms of the notion of strong incorrectness as follows

$$\exp \in C^\mathcal{K}_M \Leftrightarrow \exists n \forall m > n : y_m \text{ is strongly incorrect}$$

Proposition 4.9 $\mathcal{K}^- = \mathcal{F}^+$

Proof Let $M$ be a ground model. Observe that, as $\mathcal{F}^+ = \mathcal{K}^+$, it suffices to show that for any $\sigma \in \text{Sen}(L_T)$:

i. $\mathcal{F}_M^+ (\sigma) = a \Leftrightarrow \mathcal{K}_M^- (\sigma) = a$

ii. $\mathcal{F}_M^+ (\sigma) = d \Leftrightarrow \mathcal{K}_M^- (\sigma) = d$

To prove i and ii, proceed just as in the proof of Proposition 3.11: observe that $O_M^+ \subseteq O_M^+$ and that the inverse of $O_M^+$ is equal to $C^\mathcal{K}_M$ and use these observations, together with Lemma 3.9 and Lemma 3.11 to obtain the desired result. \hfill \square

5 Concluding remarks

We presented the method of closure games, a novel game-theoretic valuation method for languages of self-referential truth. We illustrated how our two stable judgement theorems (and their corollaries) allow us to study and define 3- and 4-valued $SK$ theories of truth in a uniform manner. By doing so, the method of closure games sheds new light on $SK$ fixed points. In particular, the method gives us a great understanding of the interrelatedness of the various $SK$ fixed points which is testified, amongst others, by our characterization of the 3m- and 3b-valued versions of the minimal and maximal intrinsic fixed point by means of closure games.

In future work, we hope to show that the method of closure games is also a fruitful method to shed light on fixed points associated with other valuation schemas than the Strong Kleene one (or on “non-fixed point” theories of truth). Can we also use (a modified version of) the method of closure games to characterize the fixed
points of the Weak Kleene schema or the Supervaluation schema? Although Welch [7] characterized the minimal fixed point of the Supervaluation schema by game-theoretic means, the last question—which is about the class of all Supervaluation fixed points—is still open.

Finally, our (intuitive) assertoric interpretation of the constituent notions of the method of closure games—closure conditions as assertoric norms, game rules as assertoric rules—stems from certain philosophical intuitions concerning the notions of assertion and denial. To spell out these intuitions in any detail is far beyond the scope of this paper but to do so rigorously is ongoing work.

Notes

1. Modulo our symbolism which reflects that we interpret the semantic values (directly) in assertoric terms.

2. A game is one of perfect information when a player who is about to make his move in the game can see all the moves that have been made before.

3. For simplicity, Martin describes his game for a first order language whose connectives are—besides a truth predicate—\( \lor \), \( \neg \) and \( \exists \). For such a language, there is no need to incorporate the roles of verifier and falsifier in the game. However, Martin remarks that when the language contains, in addition, \( \land \) and \( \lor \), the games become more complicated. In a footnote, Martin indicates how the rules and winning conditions of his game have to be modified for such languages. We state the game which Martin describes in this footnote with one terminological difference: Martin speaks of a player “being responsible for a sentence” which is interchangeable with a player being the verifier of that sentence. We have chosen to use the verifier/falsifier terminology as this terminology is well-known from Hintikka’s influential work on game-theoretic semantics.

4. The fact that the falsifier has to list a sentence when \( \gamma \) is \( T(t) \) and \( t \) denotes a sentence of \( L_T \) or when \( \gamma \) is \( \neg \alpha \) is arbitrary: we could also let the verifier do the listing.

5. The actual assignment of player control to \( A \neg \alpha \), \( D \neg \alpha \), \( A T(t) \) and \( D T(t) \) was chosen for sake of symmetry only: for those sentences it does not matter whether they are controlled by player I or by player II.

6. The sign \( A \) can be taken to indicate that player I is the verifier, \( D \) that player II is the verifier.

7. The range of \( T_M \) may depend on \( M \), i.e., for some \( M \), the range of \( T_M \) may be a strict subset of \( V \).

8. In the rules for \( T \), \( \overline{\sigma} \in CTerm(L_T) \) is a closed term (quotational constant or not) that denotes \( \sigma \) in \( M \). In the rules for the quantifiers, \( \phi(x/t) \) denotes the result of the uniform replacement of variable \( x \) by constant \( t \) in \( \phi(x) \). As testified by, amongst others, the rules for the truth predicate \( T \), the assertoric rules depend on the details of sentential reference and are, accordingly, defined relative to a ground model \( M \).

9. The assertoric rules for truth testify that the set of all expansions depends on the ground model under consideration.
10. The condition that \( C^*_M \) and \( \Omega^*_M \) are non empty rules ensures that we do not have to consider the possibility that \( \mathcal{V}^*_M \) valuates all \( L_T \) sentences as \( a (\Omega^*_M = \emptyset) \) or as \( b (C^*_M = \emptyset) \), ensuring that \( \mathcal{V}^*_M \) is at least 2 valued. This feature will be convenient for the formulation of theorems that follow.

11. In fact, one can show that \( \mathcal{V}^{gr}_M \) is 3 valued for every ground model \( M \), whereas \( \mathcal{V}^\diamond_M \) is, depending on the ground model, either 3- or 4-valued.

12. The reader may verify this by considering the sentence \( I(c) = \neg T(c) \lor T(\tau) \), where \( T(\tau) \) is the Truthmaker.

13. Note: \( V_M \) does not need to be Strong Kleene.

14. A version of assertoric semantics that is closely related to the present one was defined in \([10]\) and further studied in \([9]\). However, none of these papers mentions the close relation between assertoric semantics and the method of closure games.

15. The definition of \( \text{Branch}_M \) depends on \( M \) for the same reasons as \( \text{EXP}_M \) does.

16. Note: \( \mathcal{T}^*_X \) is not a tree in the mathematical sense of this notion.

17. The notions of downwards and upwards saturation are closely related to the notions of downwards and upwards saturation as defined by Fitting \([1]\). However, an important (and the only) difference between Fitting’s notions and ours is that Fitting’s notions are defined with respect to the assertoric rules for \( L \) only, i.e., in his definition Fitting does not treat the rules for truth not on par with the other rules. Likewise, the other notions defined in this section are inspired by \([1]\) and differ from Fitting’s notions only in the aspect just indicated. For a proof of the claim that every set of \( AD \) sentences has an upwards closure, see \([1]\).

18. The notion of a branch in this setting is slightly different from our definition of a branch. In fact, we use ‘branch’ to denote what is more commonly called ‘completed branch’. Likewise, the notion of an assertoric tree differs from that of a tableau.

References


From Closure Games to Strong Kleene Truth


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